Tensor-Triangular Geometry

TBD Graduate Seminar on Topology

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In this seminar we will study tensor-triangulated categories, i.e., triangulated categories equipped with a compatible symmetric monoidal structure. Tensor-triangulated categories arise in many places in pure mathematics: Both in algebra, for example derived categories of chain complexes of modules with the tensor product symmetric monoidal structure; and in topology, for example the stable homotopy category with the smash product symmetric monoidal structure.

To a tensor-triangulated category $\mathcal{C}$, Paul Balmer [Bal05] associated a topological space $\mathrm{Spec}(\mathcal{C})$, now called the *Balmer spectrum*. He also showed that $\mathrm{Spec}(\mathcal{C})$ constitutes the universal recipient of a support theory on $\mathcal{C}$, much analogous to the universal property of the Zariski spectrum of a commutative ring[1]. In addition, the Balmer spectrum $\mathrm{Spec}(\mathcal{C})$ encodes a classification of the so-called (reduced) *tensor-triangular ideals* of $\mathcal{C}$. Roughly speaking, this classification answers the question whether an object Y of $\mathcal{C}$ can be built out of another object X using the operations encoded in a tensor-triangulated category.

It turns out that the Balmer spectrum can be computed explicitly for many important categories. In fact, several classification results obtained prior to Balmer's definition can naturally be phrased in this language, perhaps most importantly the thick subcategory theorem of Hopkins–Smith in stable homotopy theory.

The goal of the seminar is to introduce this theory and then go through tensor-triangular classification results for several stable categories from algebra and topology. In each case we will not assume much prior knowledge about the definition of the respective category but spend some time on introducing the relevant background.

Schedule

The seminar meets at 10:15-11:45 on Tuesdays in Room TBD.

Date		Topic	Speaker
April 15	1	Tensor-triangulated categories	TBD
April 22	2	The Balmer spectrum of a tt-category	TBD
April 29	3	The derived category of a noetherian commutative ring	TBD
May 6	4	Perfect complexes of coherent sheaves over a scheme I	TBD
May 13	5	Perfect complexes of coherent sheaves over a scheme II	TBD
May 20	6	The stable module category of a finite group I	TBD
May 27	7	The stable module category of a finite group II	TBD
Jun 3	8	The stable homotopy category I	TBD
Jun 17	9	The stable homotopy category II	TBD
Jun 24	10	The stable homotopy category III	TBD
Jul 1	11	The equivariant stable homotopy category I	TBD
Jul 8	12	The equivariant stable homotopy category II	TBD

Schedule of talks

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[1] Indeed tensor-triangular categories – or perhaps rather their homotopical enhancement symmetric monoidal stable ∞-categories – can be viewed as categorified commutative rings.

Syllabus

Talk 1: Tensor-triangulated categories

This talk introduces the main objects in the seminar: Triangulated categories equipped with a bi-exact symmetric monoidal product. Go through the axioms of triangulated categories and exact functors as well as some of their elementary properties. Explain (in survey style, without going through all the definitions) that the homotopy category of a (presentably) symmetric monoidal stable ∞-category and the homotopy category of a closed symmetric monoidal stable model category are tensor-triangulated categories.
References: [Bal05; Hov99], [HPS97, Appendix A], [Bal20]

Talk 2: The Balmer spectrum of a tt-category

This talk introduces the Balmer spectrum of a tt-category, following [Bal05]. Define tensor-ideals, prime tensor-ideals, the Balmer topology and some of its elementary properties from [Bal05, Section 2], in particular the contravariant functoriality in tt-functors. Introduce the notion of a support datum on a tt-category and prove the universal property ([Bal05, Thm. 3.2]). Define radical tt-ideals and discuss the proof that every tt-ideal is radical if all objects are dualizable (the definition of which you might need to recall if it has not been discussed in the first talk). Prove the classification of all tt-ideals in terms of Thomason subsets ([Bal05, Thm 4.10]), and explain the simplification to specialization closed subsets in the notherian case (Sec. 5). Since the remaineder of the seminar is concerned with studying examples, it is OK not to spend much time on these for now. If you have time you can mention the structure sheaf ([Bal05, Sec. 6]).
References: [Bal05; Bal20]

Talk 3: The derived category of a noetherian commutative ring

The goal of this talk is to compute the Balmer spectrum of the derived category of a noetherian commutative ring, following Hopkins and Neeman [Nee92]. Recall the definition of the derived category, and then go through the proof in [Nee92]. The paper is phrased in terms of the classification of all thick subcategories, but also explain that it gives a homeomorphism between the Balmer spectrum of $\mathcal{D}^b(R)$ and the Zariski spectrum of R.
References: [Nee92; Bal05; Bal20]

Talk 4: Perfect complexes on a qcqs scheme I

This is the first of two talks on Thomason's computation of the Balmer spectrum [Tho97] of perfect complexes on a qcqs scheme. Please coordinate with the second speaker on how to divide the material, but generally this talk should introduce the relevant definitions from scheme theory (in particular what 'qcqs' means) and a construction of the tt-category of perfect complexes of coherent sheaves (see the background references provided in [Tho97]).
References: [Tho97]

Talk 5: Perfect complexes on a qcqs scheme II

Building on the previous talk, go through the proof of the main result (Thm 3.15) of [Tho97], rephrased in terms of a computation of the Balmer spectrum of the tt-category of perfect complexes.
References: [Tho97; Bal05; Bal20]

Talk 6: The stable module category of a finite group I

This is the first of two talks on the computation of the Balmer spectrum [BCR97] of the stable module category of a finite group. Please coordinate with the second speaker on how to divide up the material. Introduce the stable module category (as a tt-category) as the additive quotient by projective modules, and explain the alternative description as a Verdier quotient of the derived category, see [Bal20, pp. 5.1, 5.2]. Afterwards, start working towards the proof of the main result of [BCR97], see the description for the next talk.
References: [BCR97; Bal05; Bal20]

Talk 7: The stable module category of a finite group II

Continue and finish the proof of the identification of the Balmer spectrum of the stable module category with the projective support variety in [BCR97] reformulated as in [BCR97, Thm 5.3]. See also the references in [Bal20] regarding other proofs.
References: [BCR97]

Talk 8: The stable homotopy category I

This is the first of three talks on the thick subcategory theorem of Hopkins–Smith [HS98], the computation of the Balmer spectrum of the homotopy category of finite spectra. This talk should give an introduction to the tt-category of spectra and a characterization of its compact objects as (de-)suspensions of suspension spectra of finite CW-complexes. You can choose whether you want to discuss an ∞-categorical or a model categorical approach to spectra, but the focus should be on the main properties of the stable homotopy category rather than technicalities of a model. Explain that the subcategory of compact objects is equivalent to the Spanier-Whitehead category, which is easy to define directly.

If you have time, introduce the complex bordism spectrum MU and start discussing its relation to formal group laws [Rav92, Sec. 3].
References: [HPS97], [DHS88], [Rav92]

Talk 9: The stable homotopy category II

This is the second talk on the thick subcategory theorem. Building on the first talk, continue the discussion of MU and introduce Morava K-theories. Discuss the various forms of the nilpotence theorem [Rav92, Thms. 1.4.2, 5.1.4 and 9.0.1] and its version in terms of Morava K-theories rather than MU [Rav92, Cor. 5.1.5] (see also [Lur, Lecture 25]). If you want you can try to give a summary of the proof of the nilpotence theorem ([Rav92, Sec. 9], but if you would rather use this as a blackbox and focus on the equivalence between its different versions that is also fine.
References: [DHS88; Rav92; Lur; HPS97]

Talk 10: The stable homotopy category III

Prove the thick subcategory theorem (and phrase it as a computation of the Balmer spectrum of finite spectra) using the nilpotence theorem, following [HPS97, Sec. 5.2] (also see [Rav92, Sec. 5]).

If you have time, you can give an overview of [Rav92, Sec. 6] that the existence of v_n-self-maps is a 'thick property'.
References: [Rav92; HPS97; Bal05; Bal20]

Talk 11: The equivariant stable homotopy category I

Introduce G-spectra for a compact Lie group G by inverting representation spheres (see [Hau] and the references therein), with an emphasis on geometric fixed point functors as a family of jointly conservative symmetric monoidal functors. Discuss the compact objects.

Following [BGH20], introduce type functions and explain that they characterize tt-ideals [BGH20, Cor. 3.7 and Thm. 3.14]. Introduce the Hausdorff topology on the set of conjugacy classes of closed subgroups and explain that the Balmer topology is determined by this topology and the inclusions between primes [BGH20, Cor. 5.8]. Explain that the inclusion between primes can be interpreted as a chromatic version of classical Smith theory [Hau, Sec. 4].
References: [BGH20; KL24; Hau]

Talk 12: The equivariant stable homotopy category II

The goal of this final talk is to determine the inclusions between the Balmer primes for abelian groups. Explain the main technique (cf. [Hau, Prop. 4.6] and the references therein) for showing the existence of inclusions via computing the height shift of Tate constructions. It is probably too much to explain this computation in general (the simplest proof is probably [BK24], but it would be good to see the computation for equivariant K-theory [Hau, Example 4.9].

For the proof of non-inclusions, or in other words the proof of the existence of finite G-spectra with large height shift between their fixed points: Give a summary of the construction of Kuhn–Lloyd [KL24] via summands of smash powers of equivariant projective and lens spaces, see also [Hau, Thm 4.12 and surrounding paragraphs].
References: [Hau; BK24; BGH20; Bal20]

References

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