PRISMATISATION OVER k

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1 Recollections

Let us begin with a recollection on some of what has been covered so far. The outline of the cohomology-theories-via-ring-stacks programme looks like this. Let R be any base ring.

- Consider a functor from commutative rings to (animated) R-algebras, suggestively denoted $\mathbb{G}_a^\star$.
- Any R-scheme X (albeit frequently equipped with a smoothness condition if we want to relate back to a definition in terms of differential information) can then be transmuted to a stack $X^\star$ using the functor of points
$$X^\star(A) = \mathrm{Map}(\mathrm{Spec}(\mathbb{G}_a^\star(A)), X) = X(\mathbb{G}_a^\star(A)).$$
- The assignment $X \mapsto \mathrm{R}\Gamma_\star(X) := \mathrm{R}\Gamma(X^\star; \mathcal{O}_{X^\star})$ defines a cohomology theory on R-schemes.
- The natural category of coefficients for X is then $\mathrm{QCoh}(X^\star)$. Given a quasicoherent sheaf $\mathcal{E}$ on $X^\star$ one may then define
$$\mathrm{R}\Gamma_\star(X; \mathcal{E}) := \mathrm{R}\Gamma(X^\star; \mathcal{E}).$$

The key examples for us were as follows.

i) *de Rham cohomology in characteristic zero* is given by the ring stack on $\mathbb{Q}$-schemes
$$\mathbb{G}_a^{\mathrm{dR}} = \mathbb{G}_a/\widehat{\mathbb{G}}_a$$

ii) *Hodge-filtered de Rham cohomology in characteristic zero* is given the ring stack on $\mathbb{A}^1/\mathbb{G}_m$-schemes
$$\mathbb{G}_a^{\mathrm{dR},+} = \mathrm{cone}(\widehat{\mathbb{V}}(\mathcal{O}(-1)) \xrightarrow{t} \mathbb{G}_{a,\mathbb{A}^1/\mathbb{G}_m})$$
with generic fibre $\mathbb{G}_a^{\mathrm{dR}}$ and special fibre $\mathbb{G}_a^{\mathrm{Hod}} \simeq \mathbb{G}_{a,\mathrm{B}\mathbb{G}_m} \oplus \widehat{\mathbb{V}}(\mathcal{O}(-1))[1]$.

iii) de Rham cohomology of p-adic formal schemes is given by the ring stack on V-schemes[1]

$$\mathbb{G}_a^{\mathrm{dR}} = \mathbb{G}_a/\mathbb{G}_a^\sharp \simeq \mathrm{cone}(p :: F_*W \longrightarrow F_*W).$$

iv) Hodge-filtered de Rham cohomology of p-adic formal schemes is given by the ring stack on $\mathbb{A}^1/\mathbb{G}_m$-schemes

$$\mathbb{G}_a^{\mathrm{dR},+} = \mathrm{cone}(\mathbb{V}(\mathcal{O}(-1))^\sharp \xrightarrow{t} \mathbb{G}_{a,\mathbb{A}^1/\mathbb{G}_m})$$

with generic fibre $\mathbb{G}_a^{\mathrm{dR}}$ and special fibre $\mathbb{G}_a^{\mathrm{Hod}} \simeq \mathbb{G}_{a,\mathrm{B}\mathbb{G}_m} \oplus \mathbb{V}(\mathcal{O}(-1))^\sharp[1]$.

v) Hodge filtered conjugate filtered de Rham cohomology is given by the ring stack

$$\mathbb{G}_a^C = \mathrm{cone}(d_{u,t} \colon M_u \longrightarrow W)$$

where $d_{u,t}$ comes from the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{G}_a^\sharp \arrow[r] \arrow[d, "u^\sharp"] \arrow[dr, phantom, "\ulcorner"] & W \arrow[r, "F"] \arrow[d] & F_*W \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & \mathbb{V}(\mathcal{L})^\sharp \arrow[r] \arrow[d, "t^\sharp"] & M_u \arrow[r] \arrow[d, "d_{u,t}"] & F_*W \arrow[r] \arrow[d, "p"] & 0 \\
0 \arrow[r] & \mathbb{G}_a^\sharp \arrow[r] & W \arrow[r] & F_*W \arrow[r] & 0
\end{tikzcd}$$

over $C = \mathrm{Spec}(k[u,t]/ut)/\mathbb{G}_m$ with u in weight -1 and t in weight 1 so that it universally carries a line bundle $\mathcal{L}$ with maps

$$\mathcal{O} \xrightarrow{u} \mathcal{L} \xrightarrow{t} \mathcal{O}$$

such that $tu = 0$.

1.1 Crystalline cohomology

Let us recall a special feature of de Rham cohomology in mixed characteristic, namely that it functorially only depends on the special fibre.

Theorem 1.1.1 (Crystalline miracle). *Let V be a p-complete commutative ring which is p-torsion free and X/V a smooth p-adic formal scheme. Then the stack $(X/V)^{\mathrm{dR}}$ depends functorially on $X_{p=0}$.*

Proof. Per definition, for any p-nilpotent V-algebra R one has

$$(X/V)^{\mathrm{dR}}(R) = \mathrm{Map}_V(\mathrm{Spec}(\mathbb{G}_a^{\mathrm{dR}}(R)), X)$$

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[1] V is any p-complete commutative ring with bounded p^∞-torsion

so it suffices to show that $\mathbb{G}_a^{\mathrm{dR}}$ admits a V/p-algebra structure as the right hand side then functorially identifies with $\mathrm{Map}_{V/p}(\mathrm{Spec}(\mathbb{G}_a^{\mathrm{dR}}(R), X_{p=0})$. This follows either from the identification $\mathbb{G}_a^{\mathrm{dR}} \simeq F_*W \mathbin{/\!\!/} p$ or by concretely looking at the case $R = \mathbb{Z}_p$ where $\mathbb{G}_a^{\sharp}(R) = p\mathbb{Z}_p$. □

Accordingly, one can make the following *definition.*

Construction. Write $k = V/p$, given a smooth k-scheme Y, its crystalline stack $(Y/V)^{\mathrm{cris}}$ is then defined as the sheaf on p-nilpotent V-algebras R given by

$$(Y/V)^{\mathrm{cris}}(R) = \mathrm{Map}_k(\mathrm{Spec}(\mathbb{G}_a^{\mathrm{dR}}(R)), Y),$$

i.e. transmutation by the k-algebra stack $\mathbb{G}_a^{\mathrm{dR}}$ on p-nilpotent V-algebras.

Note the subtlety here: $\mathbb{G}_a^{\mathrm{dR}}$ and $(\mathbb{G}_a/V)^{\mathrm{cris}}$ are given by the same functor on p-nilpotent V-algebras, but the latter is regarded as landing in animated k-algebras while the former is not. Further, recall that it is the *target* category of a ring stack that becomes the base category on which one can transmute. Crystalline cohomology therefore takes as input a smooth k-scheme and outputs a stack whose coherent cohomology lives in the p-complete derived category of V, thus moving from characteristic p to mixed characteristic. As such, crystalline cohomology takes as input the p-torsion free ring V and produces a cohomology theory on $k = V/p$-schemes but it cannot be defined with just the knowledge of k, hence why V appears in the notation. We will see below that prismatic cohomology relative to the unique (crystalline) perfect prism $(W(k), (p))$ associated to k (which is equivalent to absolute prismatic cohomology of k-schemes) provides a variation on this construction which is independent of an initial choice of V.

Remark. It is of course a *theorem* that this agrees with the traditionally defined crystalline cohomology (in terms of the crystalline site or de Rham–Witt complex), but we take it as a *definition.*

The crystalline miracle can then be stated as a functorial equivalence

$$(X/V)^{\mathrm{dR}} \simeq (X_{p=0}/V)^{\mathrm{cris}}$$

for X a smooth p-adic formal scheme over V.

Remark. Suppose $V = W(k)$ with k perfect so the Witt vector Frobenius $F = W(\phi)$ comes from the Frobenius ϕ on k. Since

$$(X/W(k))^{\mathrm{dR}} \simeq (X_{p=0}/W(k))^{\mathrm{cris}}$$

functorially depends on $X_{p=0}$, the relative Frobenius

$$\phi_X\colon X_{p=0} \longrightarrow X^{(1)}_{p=0} \simeq (\phi^*X)_{p=0}$$

associated to the then induces a map on de Rham cohomology of the form

$$\phi_X\colon \phi^*\mathrm{R}\Gamma_{\mathrm{dR}}(X/W(k)) \longrightarrow \mathrm{R}\Gamma_{\mathrm{dR}}(X/W(k))$$

which we call the Crystalline Frobenius.

2 Prismatic cohomology: baby steps

Prismatic cohomology, and the ensuing notion of syntomic cohomology, arises from the need for a suitable *universal* cohomology theory on p-adic formal schemes $\mathfrak{X}$ that simultaneously knows about the crystalline cohomology of the special fibre and the étale cohomology of the generic fibre, i.e. the starting point for (integral) p-adic Hodge theory! Here's an example of a result one can prove using this.

Theorem ([BMS18, Theorem 1.1(ii)]). *Let K be a complete discretely valued nonarchimedean extension of $\mathbb{Q}_p$ with perfect residue field k and completed algebraic closure C. Let $\mathfrak{X}$ be a smooth and proper p-adic formal scheme over $\mathcal{O}_K$. Then for all $n, i \geq 0$ we have an inequality*

$$\mathrm{length}_{\mathbb{Z}_p}(\mathrm{H}^i_{\mathrm{cris}}(\mathfrak{X}_k/W(k))_{\mathrm{tor}}/p^n) \geq \mathrm{length}_{\mathbb{Z}_p}(\mathrm{H}^i_{\text{ét}}(\mathfrak{X}_C;\mathbb{Z}_p)_{\mathrm{tor}}/p^n).$$

2.1 Generalities

The input for prismatic cohomology is a prism, namely a pair (A, I) of a δ-ring A and a Cartier divisor I in $\mathrm{Spec}(A)$ satisfying the following.

i) A is (p, I)-adically complete.

ii) Let $\phi_A\colon A \to A$ denote the Frobenius lift associated to the δ-structure, i.e. $\phi_A(x) = x^p + p\delta(x)$. Then we ask that p is contained in $I + \phi_A(I)A$.

This is designed to generalise the notion of a perfect(oid) ring. Examples include.

- If A is any p-complete δ-ring, then pair $(A, I = (p))$ is the associated *crystalline prism*.
- If ϕ_A is an isomorphism, we say the prism (A, I) is perfect. The constructions $A \mapsto A/I$, $R \mapsto (\mathrm{A}_{\mathrm{inf}}(R), \ker(\theta))$ induce an equivalence of categories between perfect prisms and perfectoid rings[2].
- At the intersection of both is the crystalline perfect prism $(W(k), (p))$ associated to a perfect field k of characteristic p.

Relative prismatic cohomology is then an assignment sending a bounded p-adic formal A/I-scheme X to a stack $(X/A)^{\mathbb{\Delta}}$ over A whose coherent cohomology $\mathrm{R}\Gamma_{\mathbb{\Delta}}(X/A) \in \mathcal{D}(A)$ is the prismatic cohomology of X. This recovers the following cohomology theories, see [BS22, Theorem 1.8].

i) If $(A, (p))$ is a crystalline prism, there is an equivalence

$$\mathrm{R}\Gamma_{\mathrm{cris}}(X/A) \simeq \phi_A^* \mathrm{R}\Gamma_{\mathbb{\Delta}}(X/A).$$

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[2] Recall that $\mathrm{A}_{\mathrm{inf}}(R) = W(R^\flat)$ and $\ker(\theta)$ is a principal ideal generated by a regular element ξ such that $\delta(\xi)$ is a unit.

ii) Write $\overline{A} = A/I$ and $M\{i\} = M \otimes_{\overline{A}} (I/I^2)^{\otimes i}$, then there is a Hodge–Tate comparison

$$\Omega^i_{X/\overline{A}}\{-i\} \cong \mathrm{H}^i(\mathrm{R}\Gamma_{\Delta}(X/A) \otimes_A \overline{A})$$

for X affine.

iii) There is a de Rham comparison theorem

$$\mathrm{R}\Gamma_{\mathrm{dR}}(X/\overline{A}) \simeq \mathrm{R}\Gamma_{\Delta}(X/A) \otimes_{A,\phi_A} \overline{A}.$$

iv) If (A, I) is perfect, there is an étale comparison theorem

$$\mathrm{R}\Gamma_{\text{ét}}(X_\eta; \mathbb{Z}/p^n) \simeq (\mathrm{R}\Gamma_{\Delta}(X/A)/p^n[1/I])^{\phi=1}.$$

Instead of telling you anything at all about this theory, we will begin today by nibbling off a tiny little corner: namely the case of a crystalline perfect prism in which case we have encountered this object before.

Remark. A quick remark for the experts: there is of course also a very important notion of *absolute* prismatic cohomology, which does not depend on a base prism but instead works relative to the stack WCart (somewhat of a moduli of prisms, or pseudo-universal object in stacks). Alternatively, one can extend relative prismatic cohomology to take as input just a δ-ring in which case the absolute case is recovered as relative to $\mathbb{Z}_p$. Either way, a structural result is that if X is defined over a perfectoid ring $R \cong A/I$ then the absolute prismatisation of X agrees with its relative prismatisation relative to (A, I). Since we only work with perfect prisms below, we make no reference to the absolute theory.

2.2 Prismatisation over k

Note from the first point above that prismatic cohomology relative to a crystalline prism gives a Frobenius untwist of crystalline cohomology. For the remainder of this discussion, let us work relative to a perfect field k of characteristic p with Witt vectors $W = W(k)$.

Definition 2.2.1. $\mathbb{G}_a^{\Delta} = W /\!\!/ p$. The transmutation of a smooth k-scheme X by $\mathbb{G}_a^{\Delta}$ is denoted X^{Δ}.

Well, just that is now enough to get the entire theory off the ground as we saw before. Let us make a few first observations.

Remark. $\mathbb{G}_a^{\Delta}$ is to be regarded as a k-algebra stack on p-nilpotent W-algebras. Indeed, it lives over k as the map $W(k)/p \to k$ is an equivalence since k is perfect. Compare this to the ring stack $F_*W /\!\!/ p$ on p-nilpotent W-algebras which

- Lands in k-algebras by the argument from our crystalline miracle.

- Consequently gave rise to the theory of crystalline cohomology for smooth k-schemes.

So the Frobenius untwist could not be more obvious. Recall furthermore that we are in the perfect setting so this was not too hard to guess.

Let us now list a bunch of immediate properties of this construction. The primary thing to pay attention to here is that many pesky Frobenius twists from the de Rham/crystalline theory have now been made to disappear.

i) $\phi^* X^{\Delta} \simeq (X/W)^{\mathrm{cris}}$ where ϕ is the Witt vector Frobenius. This is immediate. In particular, we obtain[3]

$$\phi^* \mathrm{R}\Gamma_{\Delta}(X/W)/p \simeq \mathrm{R}\Gamma_{\mathrm{dR}}(X/k).$$

ii) By Frobenius untwisting, $\mathrm{R}\Gamma_{\Delta}(X/W)/p$ admits an increasing conjugate filtration which is now k-linear and has associated graded Hodge cohomology $\mathrm{R}\Gamma(X; \Omega^i_{X/k}[i])$.

iii) The absolute Frobenius on X begets a prismatic Frobenius $\phi_{X^{\Delta}} \colon X^{\Delta} \to X^{\Delta}$ living over the Frobenius $W(\phi) \colon W \to W$. On the level of ring stacks, this comes from the ϕ-semilinear endomorphism F of $W /\!\!/ p$ induced by the Witt vector Frobenius F.

The crystalline theory also comes with a gerbe, namely setting

$$X^{\mathrm{HT}} := X^{\Delta}_{p=0}$$

one obtains the Hodge–Tate stack of X over k. Using the crystalline comparison theorem and the crystalline miracle for de Rham cohomology, while Frobenius untwisting everything, there is an identification

$$X^{\mathrm{HT}} \simeq (X^{(-1)}/k)^{\mathrm{dR}}.$$

Recall the following statement from the de Rham theory.

Proposition 2.2.2. *The k-stack $(X/k)^{\mathrm{dR}}$ is a $\mathrm{B}\mathbb{V}(T_{X^{(1)}/k})^{\sharp}$-torsor over $X^{(1)}$.*

The conjugate filtration was of course obtained from the corresponding Leray filtration. We then obtain the following for the Hodge–Tate stack

Corollary 2.2.3. *The k-stack X^{HT} is a $\mathrm{B}\mathbb{V}(T_{X/k})^{\sharp}$-torsor over X.*

Let us quickly run through the construction of this torsor, i.e. the proof of the Proposition above.

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[3] Modding out by higher powers of p the right hand side of the equivalence will recover the de Rham–Witt complex of X which is traditionally used to define crystalline cohomology.

Proof. Over k, we were able to identify $\pi_0\mathbb{G}_a^{\mathrm{dR}} = F_*\mathbb{G}_a$ and $\pi_1\mathbb{G}_a^{\mathrm{dR}} = F_*\mathbb{G}_a^\sharp$ so that this ring stack can be seen as a square-zero extension. Transmuting X along the map of ring stacks (the truncation) $\mathbb{G}_a^{\mathrm{dR}} \to F_*\mathbb{G}_a$ provides the map

$$\nu\colon (X/k)^{\mathrm{dR}} \longrightarrow X^{(1)}$$

Let R be some p-nilpotent W-algebra, then $\mathbb{G}_a^{\mathrm{dR}}(R)$ is a square-zero extension of F_*R by $\mathrm{B}F_*\mathbb{G}_a^\sharp(R) \in \mathcal{D}(R)_{\geq 0}$ so the fibre of ν over some point $\eta \in X^{(1)}(R)$ is a torsor for

$$\mathrm{Der}_k(\mathcal{O}_X, \eta_*\mathrm{B}F_*\mathbb{G}_a^\sharp(R)) \simeq \mathrm{Map}_R(\eta^*\mathrm{L}_{X/k}, \mathrm{B}F_*\mathbb{G}_a^\sharp(R)).$$

Now X is assumed smooth and $\mathrm{B}\mathbb{G}_a^\sharp(R)$ is in fact 1-connective as the delooping of $\mathbb{G}_a^\sharp(R)$ so this identifies further with $\mathrm{B}(\eta^*T_{X^{(1)}/k} \otimes_R \mathbb{G}_a^\sharp(R))$ by moving over the Frobenius twist. □

An important feature of the prismatisation is that it takes a k-scheme to a W-stack with a Frobenius lift (and in fact, it is somewhat universal with this property). In particular, let us note the following easy observation whose proof strategy will be used ad nauseam

Remark. Let X be a smooth k-scheme with a lift $\widetilde{X}/W$ and Frobenius lift $\widetilde{\phi}\colon \widetilde{X} \to \widetilde{X}$ so that $\widetilde{X}$ admits the structure of a δ-scheme over W. Then any map

$$\mathrm{Spec}(R) \longrightarrow \widetilde{X}$$

over W with R a p-nilpotent W-algebra will lift through the free δ-scheme

$$\mathrm{Spec}(W(R)) \longrightarrow \widetilde{X}$$

which after reduction mod p begets a point $\mathrm{Spec}(W(R)/p) \to X$, i.e. an R-point of $X^{\mathbb{\Delta}}$. This assembles to a map $X \to X^{\mathbb{\Delta}}$ which is in fact a flat cover (but we will not discuss why this is true).

Given such a δ-lift $\widetilde{X}$ with corresponding map $\widetilde{X} \to X^{\mathbb{\Delta}}$ one can reduce mod p to obtain a flat cover

$$\widetilde{X}_{p=0} = X \longrightarrow X^{\mathrm{HT}} = X^{\mathbb{\Delta}}_{p=0}$$

which splits the Hodge–Tate gerbe. In particular, the conjugate filtration then splits as it is the corresponding Leray filtration. In particular, note that this is true Zariski-locally so that the Hodge–Tate gerbe really is a gerbe (we previously did not argue why it was Zariski–locally trivial).

Example 2.2.4. Here are some baby examples.

- When $X = \mathrm{Spec}(k)$, one has a lift $\widetilde{X} = \mathrm{Spf}(W)$ and, per construction, this induces an equivalence $\mathrm{Spf}(W) \simeq k^{\mathbb{\Delta}}$.

- Say $X = \mathbb{A}^1_k$ with its lift $\mathbb{A}^1_{\mathrm{Spf(W)}} = \mathrm{Spf}(W[x]^\wedge_p)$ and Frobenius lift induced by $x \mapsto x^p$. Then the flat cover

  $$\mathbb{A}^1_{\mathrm{Spf(W)}} \longrightarrow (\mathbb{A}^1_k)^{\mathbbl{\Delta}} \simeq \mathbb{G}_a^{\mathbbl{\Delta}}$$

  takes the form of a natural transformation id $\to W(-)/p$ on p-nilpotent W-algebras which we claim is induced by the multiplicative lift. Indeed, unraveling the construction of this flat cover one sees that it sends a class α in R to the composite

  $$W[x]^\wedge_p \xrightarrow{\psi} W(W[x]^\wedge_p) \xrightarrow{W(\alpha)} W(R)$$

  where ψ is the δ-ring co-action map which in Witt coordinates sends x to $(\delta^i(x))_i$, i.e. $(x, 0, 0, \ldots)$ which is indeed its multiplicative lift.

3 The Nygaard filtration

Prismatic cohomology comes with a whole load of extra structure inclusing the Nygaard filtration. This will be discussed in far more detail in coming talks; let us now instead discuss its characteristic properties in the crystalline setting since these set the stage for the Nygaard filtration on prismatic cohomology. For this, recall that crystalline cohomology comes with a Frobenius

$$\phi_{X/W} \colon \phi^* \mathrm{R}\Gamma_{\mathrm{cris}}(X/W) \longrightarrow \mathrm{R}\Gamma_{\mathrm{cris}}(X/W)$$

induced by the Frobenius on X. Since we are working with differential information here, the formula $\mathrm{d}f^p = pf^{p-1}\mathrm{d}f$ tells us that the Frobenius will tend to make things more divisible by p. The Nygaard filtration then precisely encodes which part of the crystalline cohomology becomes divisible by a certain power of p upon applying the crystalline Frobenius.

Theorem. *For any smooth affine k-scheme X, the crystalline Frobenius refines to a filtered map*

$$\phi^*_{X/W} \colon \mathrm{fil}^\star_{\mathcal{N}} \phi^* \mathrm{R}\Gamma_{\mathrm{cris}}(X/W) \longrightarrow p^\star \mathrm{R}\Gamma_{\mathrm{cris}}(X/W)$$

uniquely characterised by the following properties.

- *The filtration $\mathrm{fil}^\star_{\mathcal{N}} \phi^* \mathrm{R}\Gamma_{\mathrm{cris}}(X/W)$ is complete.*
- *The map on associated graded induces an equivalence*

  $$\mathrm{gr}^i_{\mathcal{N}} \phi^* \mathrm{R}\Gamma_{\mathrm{cris}}(X/W) \simeq \tau_{\geq -i} \mathrm{gr}^i_p \mathrm{R}\Gamma_{\mathrm{cris}}(X/W).$$

While we will not construct this filtration now, let us comment on why it is unique. For this, recall the Beilinson t-structure on filtered objects in $\mathcal{D}(W)$. This is defined as follows.

- $\mathrm{Fil}\,\mathcal{D}(W)_{\mathrm{B}\geq 0} = \{X_\star \mid \mathrm{gr}^i X \in \mathcal{D}(W)_{\geq -i}\}$
- $\mathrm{Fil}\,\mathcal{D}(W)_{\mathrm{B}\leq 0} = \{X_\star \mid X_i \in \mathcal{D}(W)_{\leq -i}\}$

Note the asymmetry in the definitions. This is because the t-structure is far from left complete: any constant filtration is infinitely Beilinson connective. In fact, under the stable recollement associated to the decomposition of $\mathbb{A}^1/\mathbb{G}_m$ into $\mathrm{B}\mathbb{G}_m \subset \widehat{\mathbb{A}}^1/\mathbb{G}_m$ and $\mathbb{G}_m/\mathbb{G}_m$ this is the perverse t-structure where everything on the generic fibre is infinitely connective and for the formal neighbourhood of the special fibre we use the equivalence

$$\widehat{\mathrm{Fil}}\mathcal{D}(W) \simeq \mathrm{Ch}(\mathcal{D}(W)), X_\star \longmapsto \Sigma^* \mathrm{gr}^* X$$

with the levelwise t-structure on the right hand side. It is then clear that whatever the Nygaard filtration is, the second condition guarantees that it uniquely factors as

$$\mathrm{fil}^\star_{\mathcal{N}}\phi^* \mathrm{R\Gamma}_{\mathrm{cris}}(X/W) \longrightarrow \tau_{\mathrm{B}\geq 0} p^\star \mathrm{R\Gamma}_{\mathrm{cris}}(X/W) \longrightarrow p^\star \mathrm{R\Gamma}_{\mathrm{cris}}(X/W).$$

In fact, the first map is an equivalence. Indeed, as both filtrations are complete, either by the first assumption or the observation that the crystalline cohomology lands in p-complete W-modules, it suffices to prove this on associated graded. Using the formula $\mathrm{gr}^i \tau_{\mathrm{B}\geq 0} \simeq \tau_{\geq -i}\mathrm{gr}^i$ this follow immediately from the second assumption again. In fact, note that there is an identification

$$\tau_{\geq -i}\mathrm{gr}^i_p \mathrm{R\Gamma}_{\mathrm{cris}}(X/W) \simeq \mathrm{fil}^{\mathrm{conj}}_i \mathrm{R\Gamma}(X; \Omega^\bullet_{X/k})\{i\}$$

by the crystalline miracle.

Construction. To extend to the non-affine case, note that both filtrations above define sheaves on smooth affine k-schemes with values in $\widehat{\mathrm{Fil}}\mathcal{D}(W)$ so one can just left Kan extend.

Construction. By the prismatic-crystalline comparison theorem, we therefore also obtain a complete filtration $\mathrm{fil}^\star_{\mathcal{N}}\phi^* \mathrm{R\Gamma}_{\Delta}(X/W)$ such that the prismatic Frobenius extends to a filtered map

$$\phi^*_{X/W} \colon \mathrm{fil}^\star \mathcal{N}\phi^* \mathrm{R\Gamma}_{\Delta}(X/W) \longrightarrow p^\star \mathrm{R\Gamma}_{\Delta}(X/W)$$

coming from a Beilinson connective cover.

3.1 A concrete example: the Cartier isomorphism

Let's work this out when $X = \mathrm{Spec}(R_0)$ is smooth and affine and admits a lift $\widetilde{X} = \mathrm{Spf}(R)$ with a lift of Frobenius ϕ_R (a δ-lift).

- Since there is no coherent cohomology in sight, the complex $|\Omega^\bullet_{R/W}|$ computes $\mathrm{R\Gamma}_{\mathrm{cris}}(X/W) \simeq \mathrm{R\Gamma}_{\mathrm{dR}}(\widetilde{X}/W)$.

- This complex admits a Frobenius given by applying ϕ_R to each $\Omega^i_{R/W}$.

The idea is now as follows: the Nygaard filtration will encode how divisible the image of the Frobenius becomes by some power of p. Since the Frobenius was now a mixed characteristic object (we are using the Frobenius lift ϕ_R) we are in fact allowed to formally divide by said power of p in each degree to renormalise the complex. The second defining property of the Nygaard filtration then translates to saying that our choice was optimal: we recover the Cartier isomorphism. Now consider the subcomplexes for each $i \geq 0$

$$\begin{tikzcd}
p^i\phi^*R \arrow[r] \arrow[d] & p^{i-1}\phi^*\Omega^1_{R/W} \arrow[r] \arrow[d] & \cdots \arrow[r] & p^0\phi^*\Omega^i_{R/W} \arrow[r] \arrow[d] & \phi^*\Omega^{i+1}_{R/W} \arrow[r] \arrow[d] & \cdots \\
\phi^*R \arrow[r] & \phi^*\Omega^1_{R/W} \arrow[r] & \cdots \arrow[r] & \phi^*\Omega^i_{R/W} \arrow[r] & \Omega^{i+1}_{R/W} \arrow[r] & \cdots
\end{tikzcd}$$

denoted $\mathrm{fil}^i\phi^*|\Omega^\bullet_{R/W}| \to \phi^*|\Omega^\bullet_{R/W}|$. We claim that this is the Nygaard filtration. Completeness is immediate as we work in the p-complete category. In order to construct the filtered refinement of the crystalline Frobenius, note that in degree j the map

$$\phi^j_{X/W} \colon \phi^*\Omega^j_{X/W} \longrightarrow \Omega^j_{X/W}$$

lands in $p^j\Omega^i_{X/W}$ as it sends $\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_j$ to $\mathrm{d}f_1^p \wedge \cdots \wedge \mathrm{d}f_j^p$ which is divisible by p^j. It therefore sends $p^{i-j}\phi^*\Omega^j_{R/W}$ to $p^jp^{i-j}\Omega^j_{R/W} = p^i\Omega^j_{R/W}$ and thus induces a filtered map

$$\mathrm{fil}^\star\phi^*|\Omega^\bullet_{X/W}| \longrightarrow p^\star|\Omega^\bullet_{X/W}|.$$

Let's then check the condition on associated graded.

i) $\mathrm{gr}^i\phi^*|\Omega^\bullet_{X/W}|$ is presented by the complex

$$\phi^*R_0\{i\} \longrightarrow \phi^*\Omega^1_{R_0/k}\{i-1\} \longrightarrow \cdots \longrightarrow \phi^*\Omega^i_{R_0/k} \longrightarrow 0 \longrightarrow \cdots .$$

Note, however, that all differentials here are zero!

ii) $\mathrm{gr}^i_p|\Omega^\bullet_{X/W}|$ is presented by the complex

$$R_0\{i\} \longrightarrow \Omega^1_{R_0/k}\{i\} \longrightarrow \Omega^2_{R_0/k}\{i\} \longrightarrow \cdots$$

with the usual differentials.

We now want to show that the filtered Frobenius induces a quasi-isomorphism between these, i.e. a formality result for the latter. In degree $j \leq i$, the associated graded of the filtered crystalline Frobenius is the map (the quotient by p^j comes from undoing the twists)

$$\frac{\phi_{R/W}}{p^j} \colon \phi^*\Omega^j_{R_0/k} \longrightarrow \Omega^j_{R_0/k}$$

and for $j > i$ it is clearly zero. Now this is precisely the map inducing the Cartier isomorphism, i.e. formality of the target complex. More precisely, the map above lands in the cycles, i.e. kernel of d since $\mathrm{d}f^p = pf^{p-1}\mathrm{d}f = 0$ in characteristic p and the Cartier isomorphism tells us that the composite map (uniquely determined by sending $\mathrm{d}f$ to $[f^{p-1}\mathrm{d}f]$ for any $f \in \mathcal{O}_X$)

$$\phi^*\Omega^j_{R_0/k} \longrightarrow \mathrm{H}^j|\Omega^\bullet_{R_0/k}|$$

is an isomorphism. The point is therefore that the Nygaard filtration on the de Rham cohomology of a δ-lift of R_0 allowed us to pre-emptively divide by the correct power of p that is secretly at work in the Cartier isomorphism after reduction mod p.

Remark. Note that we really used the Frobenius lift ϕ_R in this construction. While this construction can be extended to a functorial assignment in pairs (Y, ϕ_Y) of a W-lift of some Y_0/k with a Frobenius lift, the latter is critical as it makes the conjugate spectral sequence collapse. Petrov rather recently contructed examples of smooth projective lifts Y with no Frobenius lifts for which the conjugate spectral sequence does not collapse.