Motivic Homotopy Theory

Fabio Neugebauer* Lucas Piessevaux† Qi Zhu‡

WiSe 25/26

Schedule

The seminar meets at 14:15–16:00 on Thursdays[1] in the MPIM Seminar Room.

Date		Topic	Speaker
16 Oct	1	Setting up	TBD
23 Oct	2	Slice filtration and K-theory	TBD
30 Oct	3	Six functors for SH	TBD
13 Nov	4	Transfers and motivic cohomology	TBD
20 Nov	5	The motivic Steenrod algebra	TBD
27 Nov	6	Operations in motivic cohomology	TBD
04 Dec	7	Algebraic cobordism and LEFT	TBD
11 Dec	8	Cohomology of algebraic cobordism	TBD
18 Dec	9	The Hopkins–Morel equivalence	TBD
08 Jan	10	Motivic spectra	TBD
15 Jan	11	Moduli of vector bundles	TBD
22 Jan	12	Projective bundle formula	TBD
29 Jan	13	Universality of K-theory	TBD

Schedule of talks

Overview

The goal of motivic homotopy theory, as introduced by Morel and Voevodsky, is to bring homotopical techniques into the world of algebraic geometry. The fundamental idea is to replace manifolds by smooth schemes over a base, so that the affine line $\mathbb{A}^1$ plays the role of the interval in usual homotopy theory. We aim to give the participant a feel for this category by first discussing Hoyois' proof of the Hopkins–Morel isomorphism. This passes through several motivic versions of fundamental homotopical constructions such as the identification of the Steenrod algebra for mod p cohomology and the Landweber exact functor theorem, and provides a strong connection between algebraic cobordism and motivic cohomology[2]. In the second half of the seminar[3], we discuss a more recent take on the theory of motivic spectra that in fact does away with the $\mathbb{A}^1$-homotopy invariance entirely. This compromise is motivated by compatibility with algebraic K-theory, and we will see how to set up a motivic analogue of Snaith's theorem that provides a universal property for algebraic K-theory over all base schemes, while passing through a discussion of orientations in this new setting.

*s12kneug@uni-bonn.de
†lucas@math.uni-bonn.de
‡qzhu@mpim-bonn.mpg.de

[1] No seminar on 06 Nov due to NRW topology meeting.

[2] This result has received renewed attention in recent years: over a base field of positive characteritic p we only understand the motivic Steenrod algebra (and therefore also algebraic cobordism) after inverting p. It is not known how to complete this picture to the characteristic (see [AE25; CF25] for recent innovations in this direction, and cf. the second half of the syllabus for the refined context in which op. cit. plays out), but it should be the foundational computation for setting up the hypothetical prismatic stable homotopy category.

[3] We closely follow these lecture notes and strongly recommend that the speaker use these as a main reference for the structure of the talk due to the fact that some results are improved and re-proved between references.

Syllabus

Talk 1: Setting up

Define the categories of motivic spaces and motivic spectra over a scheme in terms of homotopy-invariant Nisnevich sheaves. Mention the connection between motivic stable stems over a field and Milnor–Witt K-theory. Set up the homotopy t-structure as well as the theory of strictly homotopy-invariant sheaves. As an important example for motivic spectra define the motivic spectrum of algebraic cobordism.
References: [Bac21, Sections 2-3], [Hoy15, Section 2] [PPR08, Section 2.1]

Talk 2: Slice filtration and algebraic K-theory

Discuss the representability of (homotopy) K-theory and proceed to set up the slice filtration and motivic cohomology.
References: [Bac21, Section 4]

Talk 3: Six functors for stable motivic homotopy theory

Discuss the six-functor-formalism for motivic spectra, working up to [Hoy17, Theorem 1.1, Theorem 1.5].
References: [Hoy17], but set G to be the trivial group!

Talks 4: Transfers and motivic cohomology

Define motivic spaces and spectra with transfers and relate this to motivic cohomology and the construction of motivic Eilenberg–MacLane spaces.
References: [Hoy15, Section 4]

Talks 5: The motivic Steenrod algebra

This talk should be prepared in tandem with the next speaker since there is a significant overlap in content (and this is the technical bit!). Discuss the computation of the motivic Steenrod algebra in positive characteristic. Give a brief overview of how the classical argument of [Voe10] works in characteristic zero, and what needs to be adapted to work in positive characteristic. You can blackbox any details regarding the ℓdh topology.
References: [HKØ17]

Talk 6: Operations in motivic cohomology

In tandem with the previous speaker, continue the discussion on the motivic Steenrod algebra.
References: [Hoy15, Section 5]

Talk 7: Algebraic cobordism and Landweber exact cohomology theories

Recall the motivic spectrum of algebraic cobordism from Talk 1, discuss its relation to formal groups and set up the motivic Landweber exact functor theorem. Discuss the computation of algebraic cobordism of Graßmannians, which is very similar to the topological story.
References: [NSØ09, Sections 5-7, 9]

Talk 8: Cohomology of algebraic cobordism

Revisiting the motivic spectrum of algebraic cobordism discuss its stable path components and the cohomology of its chromatic quotients.
References: [Hoy15, Sections 3, 6]

Talk 9: The Hopkins–Morel equivalence

Finish the proof of the Hopkins–Morel equivalence and discuss some direct applications
References: [Hoy15, Sections 7-8]

Talk 10: Motivic spectra

Introduce the category of non-$\mathbb{A}^1$-invariant motivic spectra. Discuss projective and weighted homotopy invariance and highlight some of the fundamental differences with SH. Introduce the Bass fundamental Theorem in this context and proceed to prove that algebraic K-theory over arbitrary bases is representable in this category.
References: [AI22, Sections 1-2], [AHI25, Section 4], [Hoy24, Sections 1-2]

Talk 11: Moduli of vector bundles

Describe the Graßmannian model for the moduli stack of vector bundles, proceed to define oriented motivic spectra.
References: [AHI25, Section 5], [Hoy24, Section 3]

Talk 12: Projective bundle formula

Prove the projective bundle formula for oriented motivic spectra, and compute the oriented cohomology of Graßmannians. Illustrate with the example of algebraic K-theory.
References: [AI22, Sections 3-4], [AHI25, Section 6], [Hoy24, Section 4]

Talk 13: Universality of K-theory

Prove the universal property of algebraic K-theory and Selmer K-theory.
References: [AI22, Section 5], [Hoy24, Section 5]

References

[AE25] Toni Annala and Elden Elmanto. "Motivic Steenrod operations at the characteristic via infinite ramification". *arXiv preprint arXiv:2506.05585* (2025).

[AHI25] Toni Annala, Marc Hoyois, and Ryomei Iwasa. "Algebraic cobordism and a Conner–Floyd isomorphism for algebraic K-theory". *Journal of the American Mathematical Society* 38.1 (2025), pp. 243–289.

[AI22] Toni Annala and Ryomei Iwasa. "Motivic spectra and universality of K-theory". *arXiv preprint arXiv:2204.03434* (2022).

[Bac21] Tom Bachmann. *Algebraic K-theory from the viewpoint of motivic homotopy theory*. 2021. URL: https://www.mathematik.uni-muenchen.de/~bachmann/KGL-minicourse.pdf.

[CF25] Shachar Carmeli and Tony Feng. "Prismatic Steenrod operations and arithmetic duality on Brauer groups". *arXiv preprint arXiv:2507.13471* (2025).

[Hoy15] Marc Hoyois. "From algebraic cobordism to motivic cohomology". *Journal für die reine und angewandte Mathematik (Crelles Journal)* 2015.702 (2015), pp. 173–226.

[Hoy17] Marc Hoyois. "The six operations in equivariant motivic homotopy theory". *Advances in Mathematics* 305 (2017), pp. 197–279.

[Hoy24] Marc Hoyois. *Obserminar: Motivic Spectra (notes)*. 2024. URL: https://lucas-piessevaux.github.io/Motivic_Spectra_notes.pdf.

[HKØ17] Marc Hoyois, Shane Kelly, and Paul Arne Østvær. "The motivic Steenrod algebra in positive characteristic." *Journal of the European Mathematical Society (EMS Publishing)* 19.12 (2017).

[NSØ09] Niko Naumann, Markus Spitzweck, and Paul Arne Østvær. "Motivic Landweber exactness". *Documenta Mathematica* 14 (2009), pp. 551–593.

[PPR08] Ivan Panin, Konstantin Pimenov, and Oliver Röndigs. "A universality theorem for Voevodsky's algebraic cobordism spectrum". *Homology, Homotopy, and Applications* 10.2 (2008), pp. 211–226.

[Voe10] Vladimir Voevodsky. "Motivic Eilenberg-MacLane spaces". *Publications Mathématiques de l'IHÉS* 112 (2010), pp. 1–99.