The motivic filtration on TC

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Schedule

The seminar meets at 14 ct on Tuesdays in SRZ214.

Date		Topic	Speaker
	0	Introduction	Organisers
03/11	1	THH of (quasiregular semi-)perfectoids	Jacco
17/11	2	THH(BP$\langle n \rangle$) by cobar descent	Nikita
24/11	3	The even filtration and motivic filtration	Maxime
01/12	4	Unraveling Quillen–Lichtenbaum	Phil
08/12	5	Quillen–Lichtenbaum in syntomic cohomology	Thorger
15/12	6	Segal conjecture and F-smoothness for ℓ	Georg
22/12 *or* 12/01	7	Syntomic cohomology of ℓ	TBD

Table 1: Schedule of talks

Introduction

Various localising invariants of (spectral) schemes such as algebraic K-theory, topological cyclic homology, etc. are designed in order to encode arithmetic or chromatic properties of the algebraic object of interest and as such remain largely inaccessible from a computational point of view. The idea of a *motivic filtration* is to filter such invariants in a way that the associated graded may be interpreted as a corresponding *motivic cohomology theory* which then constitutes the E_2-page of a motivic spectral sequence converging to the values of the localising invariant of interest. In good cases, this associated motivic cohomology theory is more amenable to computation and various properties of the localising invariant such as its redshifting behaviour, whether it satisfies the telescope conjecture etc. are wholly detected on the level of motivic associated graded. This perspective has taken many forms in different contexts; perhaps the first formalisation of this idea appeared in work of Beĭlinson–Lichtenbaum who conjectured the existence of a bigraded motivic cohomology theory on schemes appearing as the associated graded of a putative motivic filtration on algebraic K-theory, normalised by a

set of étale and rational comparison results and vanishing conjectures. These conjectures served as the impetus for an expansive and presently very active corpus in the realm of motivic homotopy theory. In this seminar, we will focus on a small part of the story that concerns itself with TC, perhaps to be thought of as an approximation of algebraic K-theory, hence lives squarely in the étale and non-$\mathbb{A}^1$-invariant setting. Inspired by the computations of Bhatt–Morrow–Scholze in the setting of p-adic Hodge theory that gave rise to the modern construction of absolute prismatic cohomology and syntomic cohomology, Hahn–Raksit–Wilson (re)construct a motivic filtration on TC of a commutative ring that makes reference exclusively to its topological Hochschild homology viewed as an $\mathbb{E}_\infty$ algebra in cyclotomic spectra in such a way that the construction filtration only makes reference to the notion of even numbers. This observation allows one to extend the construction of this motivic filtration to the setting of $\mathbb{E}_\infty$-rings in spectra, providing a –in fact *the*– computational entryway into the chromatic behaviour of the topological cyclic homology of ring spectra. This has led to celebrated recent breakthroughs in Ausoni–Rognes' *redshift* programme due to Hahn–Wilson, Angelini-Knoll–Hahn–Wilson and others as well as the disproof of the telescope conjecture due to Burklund–Hahn–Levy–Schlank.

Overview of the seminar

As all motivic filtrations in this seminar are filtrations on TC and are built out of the *even filtration* on $\mathbb{E}_\infty$-rings, we begin in Talk I with the fundamental motivating computation of Bhatt–Morrow–Scholze: namely that the p-complete topological Hochschild homology of a quasiregular semiperfectoid ring is concentrated purely in even degrees. Talk II works in the world of ring spectra, providing another piece of historical evidence for why the motivic filtration on topological Hochschild homology ought to be built out of an even filtration: namely Hahn–Wilson's computation of topological Hochschild homology of truncated Brown–Peterson spectra by means of a resolution by suggestively even objects. Talk III lifts the veil by *constructing* the motivic filtration by means of the even filtration and showing that the two motivating examples above were in fact instances of this construction. To illustrate a primary use case of this motivic filtration, we then enter a computational mode and build up to Hahn–Wilson's proof of Quillen–Lichtenbaum in the topological cyclic homology of Brown–Peterson spectra[I]. Talk IV presents a strategy for proving Quillen–Lichtenbaum by reducing it to certain properties of a cyclotomic spectrum. Talk V presents a modern approach to the former in which these properties of a cyclotomic spectrum can be phrased entirely in terms of the algebraic properties of its motivic associated graded. In Talks VI and VII we will then focus on applying these

[1]Instead of computing the syntomic cohomology of interesting yet truncated commutative rings, we will rather compute the syntomic cohomology of a particular class of higher chromatic ring *spectra* which can be thought of as higher chromatic analogues of number rings namely the truncated Brown–Peterson spectra.

results in the special, and simpler, case of ℓ culminating in a description of the associated graded of the motivic filtration on its topological cyclic homology.

Literature

Talks II-VII cover material contained in [HW22] and [HRW22]. This syllabus above is closely inspired by that of the 2024 Talbot workshop on topological cyclic homology of ring spectra whose talk notes can be found here and should serve as a good template for talks and introduction to the material.

Talks

Talk I – THH of a (quasiregular semi-)perfectoid

Prove that p-complete $\mathrm{THH}(R)_p$ of a perfectoid ring R is concentrated in even degrees, generalising the classical case of $R = \mathbb{F}_p$ by Bökstedt. A direct algebraic proof of this result first appears as [BMS19, Theorem 7.1 (1)]. For a homotopical perspective on the same problem, you may instead present Mao's argument[2] which establishes a generalisation of the Hopkins–Mahowald theorem by exhibiting R as an $\mathbb{E}_2$ Thom spectrum over the spherical Witt vectors of its tilt: [Mao23, Theorem 1.13]. The desired evenness computation then follows swiftly, see [Mao23, Proposition 1.17]. In the remaining time, present the computation in [BMS19, Proposition 6.3] which allows us to fully compute TC^- and TP of perfectoid rings.

Talk II – THH of $\mathrm{BP}\langle n\rangle$ by cobar descent

Blackbox the MU-$\mathbb{E}_3$-structure on $\mathrm{BP}\langle n\rangle$ and compute $\mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU})$ as in [HW22, Theorem 2.5.4]. Review the decent spectral sequence as in [HW22, Appendix C] and analyze it for the map $\mathrm{THH}(\mathrm{BP}\langle n\rangle) \to \mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU})$ as in [HW22, Proposition 6.11].

Talk III – The even filtration and motivic filtrations

Define the even filtration on an $\mathbb{E}_\infty$-ring following [HRW22] and use this to construct motivic filtrations on THH of $\mathbb{E}_\infty$-rings. Discuss the notion of *eff* maps and *eff* descent for the even filtration, i.e. [HRW22, Corollary 2.2.17]. Compare the resulting motivic filtrations to the BMS filtration[3],relating back to Talk I, as in [HRW22, Theorem 5.0.3]. Further, recover the cobar complex computations[4]

[2] For a gentle introduction to the ideas of this paper, see Mao's talks slides here.

[3] You can work one prime at a time and ignore the global motivic filtrations.

[4] This comparison result takes various forms: for ℓ viewed as an $\mathbb{E}_\infty$ form of $\mathrm{BP}\langle 1\rangle$ it is literally true as stated. For arbitrary $n \geq 1$ the cobar complex can instead be shown to compute the even filtration of $\mathrm{THH}(\mathrm{BP}\langle n\rangle)$ as a *module* over the $\mathbb{E}_\infty$-ring $\mathrm{THH}(\mathrm{MU})$. The latter is then again subsumed by Pstrągowski's perfect even filtration for $\mathbb{E}_1$-rings which tells us that the

of Talk II in terms of the even filtration, i.e. [HRW22, Example 4.2.3]. Finally, mention Novikov descent as a special case of *eff* descent, [HRW22, Corollary 2.2.21], and use this to argue that all associated graded terms of motivic filtrations are naturally structured over the moduli of formal groups by means of their module structure over the even filtration of the sphere.

Talk IV – Unraveling Quillen–Lichtenbaum

Discuss [HW22, §3]. Begin with a brief recollection on cyclotomic spectra as in [HW22, §3.2], and explain how the Quillen–Lichtenbaum property for $\mathrm{TC}(R)$ can be reduced to more tractable computational statements about the cyclotomic spectrum $\mathrm{THH}(R)$. Present the proofs in [HW22, §3.3, §3.4]; the relation to algebraic K-theory proceeds via the Dundas–Goodwillie–McCarthy theorem which you should state but need not prove.

Talk V – Quillen–Lichtenbaum in syntomic cohomology

Discuss the general framework for proving Quillen–Lichtenbaum via the motivic spectral sequence[5] in [Yan25]. The first instance of this is [Yan25, Theorem 2.3] which establishes Quillen–Lichtenbaum in terms of syntomic cohomology. As another example, present [Yan25, Proposition 2.5] which allows us to verify the telescope conjecture in the presence of a vanishing line in syntomic cohomology. Conclude with a discussion of [Yan25, Proposition 3.5, Theorem 3.6] which give an algebraic criterion on a ring spectrum under which Quillen–Lichtenbaum can be unraveled even further.

Talk VI – Segal conjecture and F-smoothness for ℓ

Prove that the Segal conjecture holds for $\mathrm{THH}(\mathrm{BP}\langle n\rangle)$ in the case $n = 1$. There is a general formalism developed in [HW22, §4] that covers any value of n that requires quite some technology which you may sketch. The argument simplifies considerably when $n = 1$ as documented in [HRW22, Theorem 6.1.4] so you should present a full proof of this special case as in loc. cit. Present the computation of the motivic associated graded of $\mathrm{THH}(\ell; \mathbb{F}_p)$ as in [HRW22, Theorem 6.3.3]. State and sketch the proof of [HRW22, Theorem 6.4.1][6]. Get started on presenting the computations in [HRW22, §6.5].

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cobar complex computes the perfect even filtration on the $\mathbb{E}_1$-ring $\mathrm{THH}(\mathrm{BP}\langle n\rangle)$ and is therefore independent of any higher multiplicative structure or MU-module structure, see [Pst25, Example 6.28]. Feel free to ignore this subtlety and focus on ℓ.

[5] Something to be appreciated here, and another *raison d'être* for the motivic approach to Quillen–Lichtenbaum is that the syntomic cohomology of a ring spectrum lives in a category, $\mathrm{gr}^{\mathrm{ev}}_*\mathbb{S}$-modules, in which Smith–Toda complexes just exist on the nose – albeit not necessarily arising as comodules associated to finite spectra. This allows us to make quantitative redshift statements without assuming the existence of a suitable Smith–Toda complex in spectra which previously inhibited uniform statements at all primes and heights.

[6] This property is analogous to the notion of F-smoothness in [BM23] hence the name.

Talk VII – Syntomic cohomology of ℓ

Finishing off the discussion of the computations in [HRW22, §6.5] from the last talk as required, proceed to compute the syntomic cohomology of ℓ as in [HRW22, §6.6]. From this, deduce Quillen–Lichtenbaum and the telescope conjecture for $\mathrm{TC}(\ell)$. Since we also presented a more general criterion for deducing these facts in Talk V, remark on how this could equally well be proved by plugging ℓ into the framework of Talk V.

References

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[Yan25] Tristan Yang. "Effective Redshift". *arXiv preprint arXiv:2505.00344* (2025).