Handout - Talk 3

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October 2024

Contents

Prelimenaries

We will begin with a list of notation to minimize confusion throughout the document:

- The smash product of spectra will be denoted by $\otimes$.
- The mapping specrum will be denoted by $\underline{\mathrm{Hom}}$.
- The set of morphism of two spectra X and Y will be denoted by $[X, Y]$.
- The Eilenberg-MacLane spectrum associated with an abelian group A will be denoted by HA.
- The HA-homology of X will be denoted by $A_*(X)$.
- The HA-cohomology of X will be denoted by $A^*(X)$.
- The mod p Steenrod algebra will be denoted by $\mathcal{A}_p$.
- The p-completion of a spectrum will be denoted by $X_p^\wedge$.
- The p-adic integers will be denoted by $\mathbb{Z}_p$.
- The dual of a morphism f will be denoted by $f^\vee$.
- The morphism induced on homotopy groups by f will be denoted by f_*.
- The morphism induced on homology groups by f will be denoted by f_*.
- The morphism induced on cohomology groups by f will be denoted by f^*.

Introduction

The plan is as follows. We'll start by constructing the classical Adams spectral sequence, providing most of the proofs and many details along the way. To do this, we will introduce Adams resolutions and demonstrate how they lead to an exact couple. From standard homological algebra, it follows that this construction yields a spectral sequence. Afterward, we'll identify the E_2-page of the spectral sequence and establish conditional convergence. We'll also briefly discuss the naturality of this spectral sequence.

In the second part of the talk, we will generalize the classical Adams spectral sequence to a more general version that involves E-homology for some spectrum E satisfying specific assumptions. Since the construction closely mirrors the classical case, we will provide more of a sketch, omitting some details to focus on the key differences from the classical sequence.

The whole document is a combination of the two sources [1] and [2].

The Classical Case

Theorem 1 (Adams Spectral Sequence)**.** *Let X be a spectrum with $\pi_*(X)$ bounded below and $(\mathbb{F}_p)_*(X)$ of finite type. Then there is a spectral sequence $E_r^{*,*}$ with differentials $d_r \colon E_r^{s,t} \to E_r^{s+r,t+r-1}$, such that*

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{A}_p}((\mathbb{F}_p)^*(X), \mathbb{F}_p) \implies \pi_{t-s}(X_p^\wedge).$$

Remark. For clarity, note that by "finite type," we mean as a graded module, i.e., finitely many generators in each dimension. Also, $\mathrm{Ext}^{s,t}_{\mathcal{A}_p}((\mathbb{F}_p)^*(X), \mathbb{F}_p) = \mathrm{Ext}^{s}_{\mathcal{A}_p}((\mathbb{F}_p)^t(X), \mathbb{F}_p)$.

Remark. Note that we are particularly interested in the case where $X = \mathbb{S}$ and $p = 2$, in which case the theorem gives the following spectral sequence:

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{A}_2}(\mathbb{F}_2, \mathbb{F}_2) \implies \pi_{t-s}(\mathbb{S})_2^\wedge,$$

where we use the fact that $(\mathbb{F}_2)^*(\mathbb{S}) = \pi_{-*}(\underline{\mathrm{Hom}}(\mathbb{S}, H\mathbb{F}_2)) = \pi_*(H\mathbb{F}_2) = \mathbb{F}_2$, and that $\pi_*(X_p^\wedge) = \pi_*(X) \otimes \mathbb{Z}_p = \pi_*(X)_p^\wedge$ under certain conditions on X, which are conveniently satisfied by $\mathbb{S}$.

Remark. In general, we often assume the condition that makes $\pi_*(X_p^\wedge) = \pi_*(X) \otimes \mathbb{Z}_p = \pi_*(X)_p^\wedge$ possible, and construct a spectral sequence converging to $\pi_*(X) \otimes \mathbb{Z}_p$. This will be discussed in more detail in a later talk.

Before jumping into the construction we will collect a few technical facts that we will need throughout this section in a Proposition.

Proposition 2. *Let X be a spectrum with $\pi_*(X)$ bounded below and $(\mathbb{F}_p)_*(X)$ of finite type. Then:*

(i) $(\mathbb{F}_p)^*(H\mathbb{F}_p) = \mathcal{A}_p$.

(ii) If K is a direct sum of suspensions of $H\mathbb{F}_p$ bounded below and of finite type, then $\pi_*(K)$ is a graded $\mathbb{F}_p$-vector space with one generator for each summand of K. More precisely, $\pi_*(K) = \mathrm{Hom}_{\mathcal{A}_p}((\mathbb{F}_p)^*(K), \mathbb{F}_p)$.

(iii) A map from X to K is equivalent to a collection of elements in $(\mathbb{F}_p)^*(X)$ that is bounded below and of finite type in the appropriate dimensions.

(iv) If a collection of elements in $(\mathbb{F}_p)^*(X)$ that is bounded below and of finite type generates it as an $\mathcal{A}_p$-module, then the corresponding map $f \colon X \to K$ induces a surjection in cohomology.

(v) $X \otimes H\mathbb{F}_p$ is a direct sum of suspensions of $H\mathbb{F}_p$, with one summand for each $\mathbb{F}_p$ generator of $(\mathbb{F}_p)^*(X)$. $(\mathbb{F}_p)^*(X \otimes H\mathbb{F}_p) = (\mathbb{F}_p)^*(X) \otimes \mathcal{A}_p$, and the map $f \colon X = X \otimes \mathbb{S} \xrightarrow{X \otimes \eta} X \otimes H\mathbb{F}_p$ induces the $\mathcal{A}_p$-module structure $(\mathbb{F}_p)^*(X) \otimes \mathcal{A}_p \to (\mathbb{F}_p)^*(X)$ in cohomology. In particular, f^* is surjective.

Proof.

(*i*) Follows from previous talks.

(*ii*) We can write $K = \bigoplus_i H\mathbb{F}_p[n_i]$ by assumption. Then $\pi_*(K) = \pi_*(\bigoplus_i H\mathbb{F}_p[n_i]) = \bigoplus_i \pi_*(H\mathbb{F}_p[n_i]) = \bigoplus_i \mathbb{F}_p$, where the i-th summand in the latest sum lies in degree n_i. Thus, $\pi_*(K)$ is indeed a graded $\mathbb{F}_p$-vector space. The latter identification becomes clear since we have $(\mathbb{F}_p)^*(K) = \pi_{-*}(\underline{\mathrm{Hom}}(K, H\mathbb{F}_p)) = [\mathbb{S}[-*], \underline{\mathrm{Hom}}(K, H\mathbb{F}_p)] = [K, H\mathbb{F}_p[*]] = [\bigoplus_i H\mathbb{F}_p[n_i], H\mathbb{F}_p[*]] = \bigoplus_i [H\mathbb{F}_p[n_i], H\mathbb{F}_p[*]] = \bigoplus_i \mathcal{A}_p$, with the i-th summand in degree n_i.

(*iii*) Notice that $(\mathbb{F}_p)^*(X) = \pi_{-*}(\underline{\mathrm{Hom}}(X, H\mathbb{F}_p)) = [\mathbb{S}[-*], \underline{\mathrm{Hom}}(X, H\mathbb{F}_p)] = [X, H\mathbb{F}_p[*]]$. This shows that we can understand elements of the mod (p) cohomology of X as maps $X \to H\mathbb{F}_p[*]$. The collection of elements $X \to H\mathbb{F}_p[n_i]$ assembles into a map $X \to \prod_i H\mathbb{F}_p[n_i]$. Now, recall that we always have a comparison map between the coproduct and product in spectra. Furthermore, this map is an isomorphism if:

(a) the family of spectra over which we take the coproduct/product is finite.

(b) the family of spectra over which we take the coproduct/product consists of Eilenberg-MacLane spectra in different dimensions.

These two facts together imply that we have $\prod_i H\mathbb{F}_p[n_i] = \bigoplus_i H\mathbb{F}_p[n_i] = K$, and we obtain a map $X \to K$. The other direction is similar. Furthermore, if the map $f \colon X \to K$ corresponds to the collection $\{f_i \in (\mathbb{F}_p)^*(X)\}$, then the induced map $f^* \colon (\mathbb{F}_p)^*(K) \to (\mathbb{F}_p)^*(X)$ sends the i-th generator of $\bigoplus_i \mathcal{A}_p$ to f_i.

(*iv*) This follows directly from the above description of the correspondence.

(*v*) We have that $(\mathbb{F}_p)_*(X) = \pi_*(X \otimes H\mathbb{F}_p)$ is a graded $\mathbb{F}_p$-vector space. Hence, we can pick a basis $\{f_i\}$. Each f_i can be represented by a map $\mathbb{S}[n_i] \to X \otimes H\mathbb{F}_p$. Tensoring with the identity of $H\mathbb{F}_p$ yields a map $H\mathbb{F}_p[n_i] \to X \otimes H\mathbb{F}_p \otimes H\mathbb{F}_p$, and composing with the multiplication of the homotopy ring spectrum $H\mathbb{F}_p$ yields a map $H\mathbb{F}_p[n_i] \to X \otimes H\mathbb{F}_p$. These maps assemble into a map $\bigoplus_i H\mathbb{F}_p[n_i] \to X \otimes H\mathbb{F}_p$. On homotopy groups, this maps the 1 in $\pi_*(\bigoplus_i H\mathbb{F}_p[n_i])$ corresponding to $H\mathbb{F}_p$ in degree n_i to f_i and is thus an isomorphism, and by Whitehead's theorem for spectra, the claim follows. The second part follows from Hurewicz for spectra and point (*iv*).

□

Adams Resolution

After stating the main theorem, we will begin constructing the spectral sequence. To do this, we will introduce Adams resolutions.

Definition 3. Let X be a spectrum with $\pi_*(X)$ bounded below and $(\mathbb{F}_p)_*(X)$ of finite type. A mod (p) Adams resolution for X is a diagram of spectra as follows:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & X_2 & \xrightarrow{i_1} & X_1 & \xrightarrow{i_0} & X_0 & = & X \\
 & & \downarrow{\scriptstyle j_2} & & \downarrow{\scriptstyle j_1} & & \downarrow{\scriptstyle j_0} & & \\
 & & K_2 & & K_1 & & K_0 & &
\end{array}$$

where each K_s is a direct sum of suspensions of $H\mathbb{F}_p$ that is bounded below and of finite type, $(j_s)^*$ is surjective, and X_{s+1} is the fiber of j_s.

Lemma 4. *For a spectrum X such that $\pi_*(X)$ is bounded below and $(\mathbb{F}_p)^*(X)$ is of finite type, an Adams resolution exists.*

Proof. Suppose we have constructed the diagram up to X_s, i.e., we have the diagram:

$$\begin{array}{ccccc} X_s & \xrightarrow{i_{s-1}} & X_{s-1} & \xrightarrow{i_{s-2}} & X_{s-2} \cdots\cdots\cdots \\ & & \downarrow{\scriptstyle j_{s-1}} & & \downarrow{\scriptstyle j_{s-2}} \\ & & K_{s-1} & & K_{s-2} \end{array}$$

Define $K_s = X_s \otimes H\mathbb{F}_p$. Then, Proposition 2 ($v$) implies that K_s is a direct sum of shifts of $H\mathbb{F}_p$ that is bounded below and of finite type (this follows from the assumption that the mod p homology is of finite type). Define j_s as the map $f\colon X = X \otimes \mathbb{S} \xrightarrow{X\otimes\eta} X \otimes H\mathbb{F}_p$, where η denotes the unit of the homotopy ring spectrum $H\mathbb{F}_p$. Proposition 2 (v) shows that $(j_s)^*$ is surjective. Now, define X_{s+1} as the fiber of j_s and continue inductively. □

Remark. Using the long exact sequence in homotopy groups, one can verify that $\pi_*(X_s)$ is bounded below for all s. Additionally, using the long exact sequence in mod (p) homology, one can verify that $(\mathbb{F}_p)_*(X_s)$ is of finite type for each s.

Construction

Suppose that X is a spectrum as above and that we have constructed an Adams resolution for X. Now we will construct a spectral sequence from this Adams resolution. For this, notice that we have long exact sequences:

$$\pi_*(X_{s+1}) \xrightarrow{(i_s)_*} \pi_*(X_s) \xrightarrow{(j_s)_*} \pi_*(K_s) \xrightarrow{\partial_{s,*}} \pi_*(X_{s+1})$$

for each s from the fibrations $X_{s+1} \to X_s \to K_s$. Now define bigraded groups D_1 and E_1 through $D_1^{s,t} = \pi_{t-s}(X_s)$ and $E_1^{s,t} = \pi_{t-s}(K_s)$, respectively. Then the long exact sequences translate to a diagram:

$$D_1 \xrightarrow{i_1} D_1 \xrightarrow{j_1} E_1 \xrightarrow{k_1} D_1$$

$$\begin{aligned} i_1 &= (i_s)_*\colon D_1^{s+1,t+1} \to D_1^{s,t} \\ j_1 &= (j_s)_*\colon D_1^{s,t} \to E_1^{s,t} \\ k_1 &= \partial_{s,t-s}\colon E_1^{s,t} \to D_1^{s+1,t} \end{aligned}$$

Notice that the exactness of the first diagrams translates to:

$$\ker i_1 = \operatorname{im} k_1, \quad \ker j_1 = \operatorname{im} i_1, \quad \ker k_1 = \operatorname{im} j_1.$$

Hence, we have derived an exact couple from the Adams resolution. Constructing a spectral sequence from this is standard homological algebra, which can be found in many books, or alternatively in the Algebraic Topology 1 notes by Hausmann. Thus, we obtain a spectral sequence $\{E_r, d_r\}$, where $d_r\colon E_r^{s,t} \to E_r^{s+r,t+r-1}$ (i.e. the differentials are of degree $(r, r-1)$).

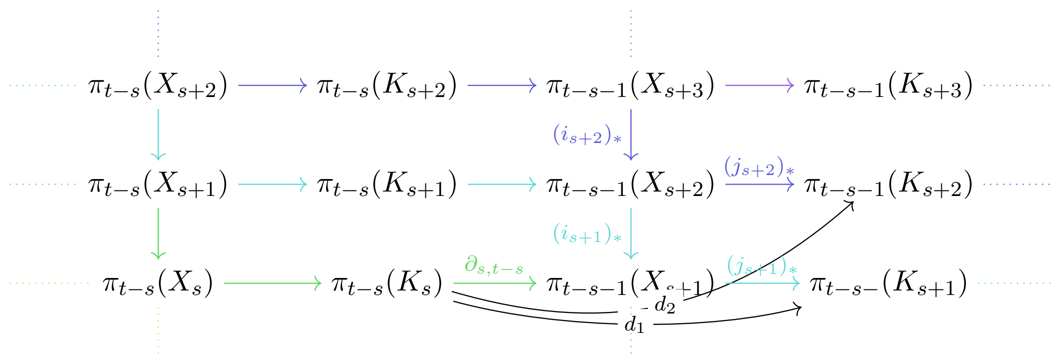
The expected abutment is the graded abelian group $\pi_*(X)$ filtered by the image groups:

$$F_*^s = \operatorname{im}(i^s\colon \pi_*(X_s) \to \pi_*(X)),$$

where by abutment we mean the expected form of the E_∞-page. This will be made more precise later on. This concludes the existence part of Theorem 1. It remains to identify the E_2-page and show some form of convergence.

Differentials

Since we will need this later on, let us give a quick description of what the differentials will look like. Consider the following diagram:



The dark blue arrows form an exact sequence induced by the fibration $X_{s+3} \to X_{s+2} \to K_{s+2}$, the light blue arrows form an exact sequence induced by the fibration $X_{s+2} \to X_{s+1} \to K_{s+1}$, and so on. By analyzing the E_1-page given from the exact couple, one can see that the $d_1^{s,t}$-morphism is given by $(j_{s+1})_* \circ \partial_{s,t-s}$. By definition, we have

$$E_2^{s,t} = \ker d_1^{s,t} / \operatorname{im} d_1^{s,t}.$$

Now suppose that an element of $E_2^{s,t}$ can be represented by $x \in \pi_{t-s}(K_s)$. Then we must have $d_1^{s,t}(x) = 0$, and hence we can lift $\partial_{s,t-s}(x)$ to $\pi_{t-s-1}(X_{s+2})$ along $(i_{s+1})_*$; i.e., we find $y \in \pi_{t-s-1}(X_{s+2})$ such that

$$(i_{s+1})_*(y) = \partial_{s,t-s}(x).$$

Now, $d_2^{s,t}(x)$ is given by $(j_{s+2})_*(y)$. This is clearly a d_1-cycle, and it can easily be shown to be well defined by a simple diagram chase.

If we now suppose that $d_2^{s,t}(x) = 0$, i.e., x represents an element of $E_3^{s,t}$, we can lift y to $\pi_{t-s-1}(X_{s+3})$ along $(i_{s+2})_*$ since $(j_{s+2})_*(y) = 0$, and repeat the procedure to find the image of $d_3^{s,t}$.

Remark. Notice that the arising spectral sequence is first quadrant after a few suspensions. Usually, one does a coordinate shift $(s,t) \mapsto (t-s, s)$ to the so-called Adams grading, in which case all non-zero entries are over a line of slope 1. The degree of the differentials becomes $(-1, r)$.

The E_2-page

In this subsection, we are going to identify the E_2-page of the spectral sequence constructed above. This will be done in two parts. First, we will show that from an Adams resolution of X, we obtain a free $\mathcal{A}_p$-resolution of $(\mathbb{F}_p)^*(X)$. Then we will use that to demonstrate that the E_1-page forms a complex that computes the relevant Ext term.

Lemma 5. *For any Adams resolution of X as above, the diagram*

$$\cdots \longrightarrow (\mathbb{F}_p)^*(\Sigma^2 K_2) \xrightarrow{\partial_2} (\mathbb{F}_p)^*(\Sigma K_1) \xrightarrow{\partial_1} (\mathbb{F}_p)^*(K_0) \xrightarrow{\varepsilon} (\mathbb{F}_p)^*(X)$$

is a free $\mathcal{A}_p$-resolution of $(\mathbb{F}_p)^*(X)$, where each of the modules is bounded below and of finite type. Furthermore, we have $\partial_s = \partial^*(j_s)^*$.

Proof. By definition, the maps $j_s : X_s \to K_s$ induce surjective maps on mod (p) cohomology. This implies that the long exact sequences on mod (p) cohomology, induced by the fiber sequences $X_{s+1} \to X_s \to K_s$, break into short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (\mathbb{F}_p)^{*-s-1}(X_{s+1}) & \xrightarrow{\partial^*} & (\mathbb{F}_p)^{*-s}(K_s) & \xrightarrow{(j_s)^*} & (\mathbb{F}_p)^{*-s}(X_s) & \longrightarrow & 0 \\
 & & \| & & \| & & \| & & \\
0 & \longrightarrow & (\mathbb{F}_p)^*(\Sigma^{s+1} X_{s+1}) & \xrightarrow{\partial^*} & (\mathbb{F}_p)^*(\Sigma^s K_s) & \xrightarrow{(j_s)^*} & (\mathbb{F}_p)^*(\Sigma^s X_s) & \longrightarrow & 0
\end{array}$$

for each $s, *$. These splice together to form a long exact sequence

$$\begin{array}{ccccccccc}
 & & (\mathbb{F}_p)^*(\Sigma^2 X_2) & & & & (\mathbb{F}_p)^*(\Sigma X_1) & & & & (\mathbb{F}_p)^*(X) \\
 & \overset{(j_2)^*}{\nearrow} & & \overset{\partial^*}{\searrow} & & \overset{(j_1)^*}{\nearrow} & & \overset{\partial^*}{\searrow} & & \overset{(j_0)^*}{\nearrow} & \\
\cdots (\mathbb{F}_p)^*(\Sigma^2 K_2) & & \xrightarrow{\qquad \partial_2 \qquad} & & (\mathbb{F}_p)^*(\Sigma K_1) & & \xrightarrow{\qquad \partial_1 \qquad} & & (\mathbb{F}_p)^*(K_0) & &
\end{array}$$

It remains to show that $(\mathbb{F}_p)^*(K_s)$ is a free $\mathcal{A}_p$-module, but we saw this in the proof of Proposition 2 (ii). Thus, the long exact sequence above does indeed form a free $\mathcal{A}_p$-resolution of $(\mathbb{F}_p)^*(X)$. $\square$

Let us now write $P^\bullet = (\mathbb{F}_p)^*(\Sigma^\bullet K_\bullet)$ to clarify that $P^\bullet \xrightarrow{\varepsilon} (\mathbb{F}_p)^*(X)$ is a free $\mathcal{A}_p$-resolution. Now we are ready to prove that the E_2-page has the correct form. From the exact couple above, we see that the E_1-page of the spectral sequence takes the following form.

$$0 \longrightarrow \pi_*(K_0) \xrightarrow{(j_0\partial_{0,*})_*} \pi_*(\Sigma K_1) \xrightarrow{(j_1\partial_{1,*})_*} \pi_*(\Sigma^2 K_2) \cdots\cdots$$

Proposition 2 (ii) implies that this is isomorphic to the complex

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{A}_p}(P^0, \mathbb{F}_p) \xrightarrow{((j_0\partial_{0,*})_*)^\vee} \mathrm{Hom}_{\mathcal{A}_p}(P^1, \mathbb{F}_p) \xrightarrow{((j_1\partial_{1,*})_*)^\vee} \mathrm{Hom}_{\mathcal{A}_p}(P^2, \mathbb{F}_p) \cdots\cdots$$

Hence, the E_1-page is given by taking the resolution $P^\bullet$ and applying $\mathrm{Hom}_{\mathcal{A}_p}(-, \mathbb{F}_p)$. The E_2-page will be the cohomology of this complex, which computes $\mathrm{Ext}^{s,t}_{\mathcal{A}_p}((\mathbb{F}_p)^*(X), \mathbb{F}_p)$ by definition.

Remark. Notice that this, in particular, shows that the E_2-page of the Adams spectral sequence does not depend on the choice of resolution, while the E_1-page did.

Naturality

It is a standard result in homological algebra that free (in fact, projective suffices) resolutions are unique up to chain homotopy. This property carries over to spectral realizations; specifically, we have a similar result for Adams resolutions. In this subsection, we will state this result and derive the naturality of the Adams spectral sequence as a corollary.

Suppose we have the two Adams resolutions of the spectra X and Y which are bounded below spectra with $(\mathbb{F}_p)_*(X)$ and $(\mathbb{F}_p)_*(Y)$ of finite type:

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & X_2 & \xrightarrow{i_1} & X_1 & \xrightarrow{i_0} & X_0 & = & X \\
& \downarrow{\scriptstyle j_2} & & \downarrow{\scriptstyle j_1} & & \downarrow{\scriptstyle j_0} & & \\
& K_2 & & K_1 & & K_0 & &
\end{array}$$

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & Y_2 & \xrightarrow{i_1} & Y_1 & \xrightarrow{i_0} & Y_0 & = & Y \\
& \downarrow{\scriptstyle j_2} & & \downarrow{\scriptstyle j_1} & & \downarrow{\scriptstyle j_0} & & \\
& L_2 & & L_1 & & L_0 & &
\end{array}$$

with associated resolutions $P^\bullet \xrightarrow{\varepsilon} (\mathbb{F}_p)^*(X)$ and $Q^\bullet \xrightarrow{\varepsilon} (\mathbb{F}_p)^*(Y)$. Then we have the following result:

Theorem 6. *With the setting as above, let $f\colon X \to Y$ be any map. Then there exists a chain map $g^\bullet\colon Q^\bullet \to P^\bullet$ lifting f^*. Furthermore, there is a map of resolutions $\{f_s\colon X_s \to Y_s\}$ lifting f and realizing $g^\bullet$ in the sense that there is a homotopy commutative diagram*

$$\begin{array}{cccccc}
\cdots & X_2 & \longrightarrow & X_1 & \longrightarrow & X \\
& \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f} \\
\cdots & Y_2 & \longrightarrow & Y_1 & \longrightarrow & Y
\end{array}$$

and given any choice of commuting homotopies, the induced map of the cofibers $g_s\colon K_s \to L_s$ induces $g^s = (\Sigma^s g_s)^*\colon Q_s \to P_s$.

If $\bar{g}^\bullet$ is a second chain map lifting f^* and $\{\bar{f}_s\}$ is a map of resolutions lifting f and realizing $\bar{g}^\bullet$, then $g^\bullet$ and $\bar{g}^\bullet$ are chain homotopic, and $\{f_s\}$ and $\{\bar{f}_s\}$ are homotopic in the sense that the composites $f_s \circ i$ and $\bar{f}_s \circ i$ are homotopic for all s.

Proof. The first half of this proof is standard in homological algebra, while the second part involves lifting the constructed chain homotopies to spectral realizations. The key insight for the second part is that we have

$$[K_s, L_s] = \mathrm{Hom}_{\mathcal{A}_p}((\mathbb{F}_p)^*(L_s), (\mathbb{F}_p)^*(K_s)),$$

which allows for the proper lifts. More details can be found in [2, Theorem 4.16]. □

Corrollary 7. *Let $f\colon X \to Y$ be a map of bounded below spectra with $(\mathbb{F}_p)_*(X)$ and $(\mathbb{F}_p)_*(Y)$ of finite type. Then there is a map*

$$f_*\colon \{E_r(X), d_r\} \to \{E_r(Y), d_r\}$$

of Adams spectral sequences, given at the E_2-level by the homomorphism

$$(f^*)^*\colon \mathrm{Ext}^{s,t}_{\mathcal{A}_p}((\mathbb{F}_p)^*(X), \mathbb{F}_p) \to \mathrm{Ext}^{s,t}_{\mathcal{A}_p}((\mathbb{F}_p)^*(Y), \mathbb{F}_p)$$

with the expected abutment homomorphism $f_*\colon \pi_*(X) \to \pi_*(Y)$.

Definition 8. An element in $E_r^{s,t}$ is said to be of filtration s, total degree $t-s$, and internal degree t. An element in $F_*^s \subseteq \pi_*(X)$, i.e., in the image of $i^s\colon \pi_*(X_s) \to \pi_*(X)$, is said to have Adams filtration $\geq s$.

Theorem 9 (Filtration Theorem)**.** *Let $\{X_s\}$ be and Adams resolution of X as above. A class $f \in \pi_*(X)$ has Adams filtration $\geq s$ if and only if the representing map $f\colon \mathbb{S}[n] \to X$ can be factored as a composite of s maps*

$$\mathbb{S}[n] = Y_s \xrightarrow{z_s} Y_{s-1} \xrightarrow{z_{s-1}} \cdots \xrightarrow{z_2} Y_1 \xrightarrow{z_1} Y_0 = X$$

where z_u induces the zero map on mod (p) cohomology for all u. In particular $F_*^s \subseteq \pi_*(X)$ is independent of the choice of Adams resolution.

Proof. Suppose f has Adams filtration $\geq s$. Then, there exists a lift $g\colon \mathbb{S}[n] \to X_s$ such that $i^s \circ g = f$. We define $Y_u = X_u$ and $z_u = i_{u-1}$ for $1 \leq u \leq s-1$, and let $z_s = i_{s-1} \circ g$.

Conversely, given a factorization of $f = z_1 \circ \cdots \circ z_s$ as above, we define $f^0\colon X \to X$ to be the identity. We then inductively find lifts $f^u\colon Y_u \to X_u$ such that the following diagram commutes:

$$\begin{array}{ccccccc}
Y_s & \xrightarrow{z_s} & Y_{s-1} & \cdots\cdots & Y_1 & \xrightarrow{z_1} & X \\
\downarrow{\scriptstyle f^s} & & \downarrow{\scriptstyle f^{s-1}} & & \downarrow{\scriptstyle f^1} & & \downarrow \\
X_s & \xrightarrow[i_{s-1}]{} & X_{s-1} & \cdots\cdots & X_1 & \xrightarrow[i_0]{} & X
\end{array}$$

The diagram commutes because the obstruction to lifting $f^{u-1} \circ z_u\colon Y_u \to X_{u-1}$ over $i_{u-1}\colon X_u \to X_{u-1}$ is given by the homotopy class of the composite $j \circ f^{u-1} \circ z_u$. This obstruction is zero since $(z_u)^* = 0$.

Let $g = f^s\colon \mathbb{S}[n] \to X_s$. Then we have $i^s \circ g = f$, which implies that f is of Adams filtration $\geq s$. □

Convergence

In this talk, we will only prove conditional convergence, the meaning of which will become clear shortly. In the following discussions, we will establish convergence to the p-completion of the homotopy of X based on this result. To do so, we first need to define our E_∞-page i.e., show that the $E_r^{s,t}$ become stable for large r.

By definition, we have $E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}_p}^{s,t}((\mathbb{F}_p)^*(X), \mathbb{F}_p) = 0$ for $s < 0$. Consequently, this also holds for $E_r^{s,t}$ for all r. In particular, if $s < r$, the image of d_r in $E_r^{s,t}$ is trivial, as $E_r^{s-r,t-r+1} = 0$. Thus, $E_r^{s,t}$ is a subgroup of $E_{r+1}^{s,t}$. We can then define

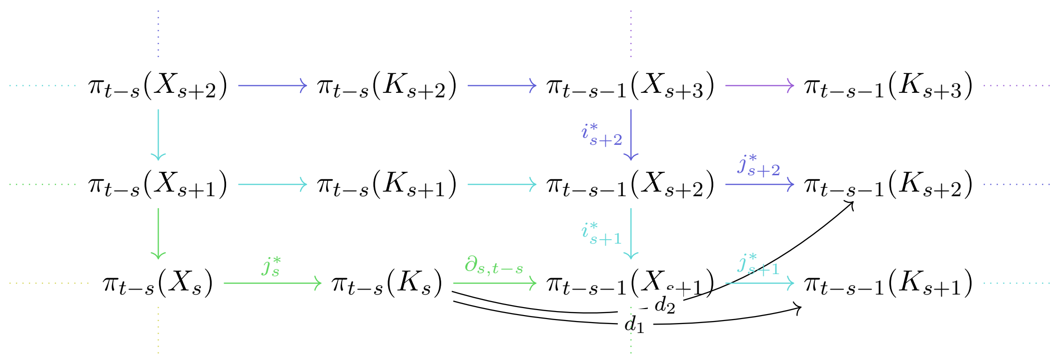
$$E_\infty^{s,t} = \bigcap_{r>s} E_r^{s,t}.$$

With that established, we are ready to state our conditional convergence result.

Theorem 10. *Let X be a spectrum as above with Adams resolution $\{X_s\}$ such that $\varprojlim X_s = *$. Then $E_\infty^{s,t}$ is the subquotient F_{t-s}^s/F_{t-s}^{s+1} of $\pi_{t-s}(X)$, and $\bigcap F_*^s = 0$, where F_*^s is defined as above.*

Proof. First, notice that since $\varprojlim X_s = *$, the Milnor $\varprojlim^1$ sequence implies that $\varprojlim \pi_*(X_s) = *$. This also implies that the intersection vanishes.

We will now use the description of the differentials we discussed earlier. Recall the diagram:



For the identification of $E_\infty^{s,t}$, let $x \in E_\infty^{s,t}$ be a nonzero class. Since $d_r(x) = 0$, we know that $\partial_{s,t-s}(x)$ can be lifted to $\pi_{t-s-1}(X_{s+r+1})$ for all r. Thus, $\partial_{s,t-s}(x) \in \varprojlim \pi_{t-s-1}(X_{s+r}) = 0$. Therefore, we conclude that $\partial_{s,t-s}(x) = 0$, and by exactness, we can write $x = (j_s)^*(y)$ for some $y \in \pi_{t-s}(X_s)$.

It suffices now to show that y has a nontrivial image in $\pi_{t-s}(X)$. Suppose, for contradiction, that it does not. Let r be the largest integer such that y has a nontrivial image $z \in \pi_{t-s}(X_{s-r+1})$. Then $z = \partial_{s-r,t-s}(w)$ for some $w \in \pi_{t-s}(K_{s-r})$, and $d_r(w) = x$, which contradicts the nontriviality of x. $\square$

Example

Let us consider a simple example. Suppose $X = H\mathbb{Z}$. The fundamental cohomology class induces a map $i\colon H\mathbb{Z} \to H\mathbb{F}_p$ with i^* being surjective. The fiber of i is also $H\mathbb{Z}$, and the inclusion map $j\colon H\mathbb{Z} \to H\mathbb{Z}$ has degree p. Therefore, we obtain an Adams resolution where $X_s = H\mathbb{Z}$ and $K_s = H\mathbb{F}_p$ for all s. The map $H\mathbb{Z} = X_s \to X_0 = H\mathbb{Z}$ has degree p^s. Consequently, we have

$$E_1^{s,t} = \begin{cases} \mathbb{F}_p & \text{if } t = s, \\ 0 & \text{if } t \neq s. \end{cases}$$

For degree reasons, there are no nontrivial differentials, so the spectral sequence collapses, yielding $E_\infty = E_1$. Hence, $E_\infty^{s,s} = \mathbb{F}_p = F_0^s / F_0^{s+1}$.

The General Case

In this section, we aim to generalize the construction from the previous section by replacing the mod (p) Eilenberg-MacLane spectrum with a more general spectrum E, where the prime example is the Brown–Peterson spectrum BP. Our goal is to replicate the previous results using E-cohomology for a given spectrum E, rather than $H\mathbb{F}_p$. Specifically, for a spectrum X, we want an E_2-page that can be computed in terms of $E^*(X)$ as a module over $E^*(E)$ and that converges to the homotopy groups of an E-local version of X in a meaningful way. Naturally, certain restrictions on E are required for this approach, which we will address here.

Experience suggests that dualizing the setup is wise; that is, we should consider $E_*(X)$ as a comodule over $E_*(E)$. With these considerations in mind, let us state the main theorem and discuss how it differs from the previous section.

Theorem 11. *Let E be an Adams-type homotopy ring spectrum and X a spectrum. Then there exists a spectral sequence $E_r^{*,*}$ with differentials $d_r\colon E_r^{s,t} \to E_r^{s+r,t+r-1}$ such that*

$$E_2^{s,t} = \operatorname{Ext}^{s,t}_{E_*(E)}(E_*(\mathbb{S}), E_*(X)) \implies \pi_{t-s}(X_E^\wedge) \coloneqq \operatorname{Tot}(X \otimes E^{\otimes(\bullet+1)}),$$

where $X_E^\wedge$ denotes the nilpotent E-completion of X.

Remark. Notice that Theorem 1 corresponds to the special case where $E = H\mathbb{F}_p$. In particular, for spectra X with $\pi_*(X)$ bounded below and $(\mathbb{F}_p)_*(X)$ of finite type, there exists an equivalence between the nilpotent $H\mathbb{F}_p$-completion and the usual p-completion of X, i.e., $X_p^\wedge \xrightarrow{\simeq} X_{H\mathbb{F}_p}^\wedge$.

Remark. In this discussion, we will not go into the precise definition of $X_E^\wedge$. It suffices to note that, in almost all cases of interest, there exists a suitable model for $X_E^\wedge$, which can be easily identified.

Adams-Type Ring Spectrum

We will briefly outline the assumptions that make E an Adams-type homotopy ring spectrum. These assumptions may initially seem somewhat mysterious but will become clearer as we proceed. Notably, these properties should be essential features of $H\mathbb{F}_p$ that allow for the constructions in the previous section.

(a) E is a homotopy-commutative ring spectrum.

(b) E is connective, meaning $\pi_r(E) = 0$ for $r < 0$.

(c) The map $\mu_*\colon \pi_0(E) \otimes \pi_0(E) \to \pi_0(E)$, induced by the multiplication μ, is an isomorphism.

(d) E is flat; that is, $E_*(E)$ is flat as a left module over $\pi_*(E)$.

(e) Let $\theta\colon \mathbb{Z} \to \pi_0(E)$ be the unique ring homomorphism, and let $R \subseteq \mathbb{Q}$ denote the largest subring to which θ extends. Then $R_*(E)$ is finitely generated over R.

These assumptions are crucial for two main purposes. First, they allow us to replicate arguments from the previous section, such as Adams resolutions and the identification of the E_2-page. Second, they ensure the existence of an E-nilpotent completion, which is essential for the convergence of the spectral sequence.

Let us discuss the conditions briefly. The first assumption is necessary in the next subsection to establish that $(\pi_*(E), E_*(E))$ forms a Hopf algebroid, which enables E_* to function as a functor to left $E_*(E)$-comodules. The fourth assumption is required for performing the homological algebra needed to describe the E_2-page. Notably, the spectral sequence would still maintain the desired convergence properties even without this fourth assumption. In this discussion, we will primarily rely on these two assumptions, while the others primarily contribute to constructing an E-nilpotent completion of a spectrum X and to the convergence proof.

Proposition 12. *The spectra MU, BP, and $H\mathbb{F}_p$ satisfy the assumptions listed above.*

Remark. The spectra $H\mathbb{Z}$, bo, and bu do not satisfy the fourth assumption. In these cases, $E \otimes E$ is not a direct sum of suspensions of E. However, they do satisfy the other four assumptions, allowing us to still obtain a convergent spectral sequence.

$E_*(E)$, $E_*(X)$, and Ext

In this subsection, we construct the necessary structure on $\pi_*(E)$, $E_*(E)$, and $E_*(X)$. Specifically, we will construct the maps that make $(\pi_*(E), E_*(E))$ into a Hopf algebroid and $E_*(X)$ into a left $E_*(E)$-comodule. This mirrors the case $E = H\mathbb{F}_p$, where $(\pi_*(H\mathbb{F}_p) = \mathbb{F}_p, (\mathbb{F}_p)_*(H\mathbb{F}_p) = \mathcal{A}_p^\vee)$ forms a Hopf algebra (i.e., a Hopf algebroid with a single object), and $(\mathbb{F}_p)^*(X)$ is an $(\mathbb{F}_p)^*(H\mathbb{F}_p) = \mathcal{A}_p$-module, or equivalently, $(\mathbb{F}_p)_*(X)$ is a $(\mathbb{F}_p)_*(H\mathbb{F}_p) = \mathcal{A}_p^\vee$-comodule.

To construct $(\pi_*(E), E_*(E))$ as a Hopf algebroid and $E_*(X)$ as an $E_*(E)$-comodule, we require the following structure maps:

- **The left unit/source map,** $\eta_L\colon \pi_*(E) \to E_*(E)$**:**
  The left $\pi_*(E)$-module structure on $E_*(E)$ (which is flat by assumption (d)) is induced by the map $E \otimes \eta\colon E = E \otimes \mathbb{S} \to E \otimes E$. We define η_L to be this map.
- **The right unit/target map,** $\eta_R\colon \pi_*(E) \to E_*(E)$**:**
  The right $\pi_*(E)$-module structure on $E_*(E)$ is induced by the map $\eta \otimes E\colon E = \mathbb{S} \otimes E \to E \otimes E$. We define η_R to be this map.
- **The coproduct/composition map,** $\Delta\colon E_*(E) \to E_*(E) \otimes_{\pi_*(E)} E_*(E)$**:**
  If we set $X = E$ in the definition of the map ψ below, we obtain $\Delta\colon E_*(E) \to E_*(E) \otimes_{\pi_*(E)} E_*(E)$.
- **The counit/identity map,** $\varepsilon\colon E_*(E) \to \pi_*(E)$**:**
  The map $\varepsilon\colon E_*(E) \to \pi_*(E)$ is induced by the multiplication map $\mu\colon E \otimes E \to E$.
- **The conjugation/inverse map,** $c\colon E_*(E) \to E_*(E)$**:**
  The conjugation map $c\colon E_*(E) \to E_*(E)$ is induced by permuting the factors in $E \otimes E \to E \otimes E$.
- **The comodule map,** $\psi\colon E_*(X) \to E_*(E) \otimes_{\pi_*(E)} E_*(X)$**:**
  The map $E \otimes \eta \otimes X\colon E \otimes X = E \otimes \mathbb{S} \otimes X \to E \otimes E \otimes X$ induces $\psi\colon E_*(X) \to \pi_*(E \otimes E \otimes X) = E_*(E) \otimes_{\pi_*(E)} E_*(X)$, where we use Lemma 14 for the last equality.

Proposition 13. *The pair $(\pi_*(E), E_*(E))$ with the structure maps described above forms a Hopf algebroid, and E-homology is a functor to the category of left $E_*(E)$-comodules, which in this case is abelian.*

Lemma 14. *The map* $(E \otimes E) \otimes (E \otimes X) = E \otimes (E \otimes E) \otimes X \xrightarrow{E \otimes \mu \otimes X} E \otimes E \otimes X$ *induces an isomorphism* $E_*(E) \otimes_{\pi_*(E)} E_*(X) \to \pi_*(E \otimes E \otimes X)$.

Proof. The result is straightforward for $X = \mathbb{S}[n]$. It follows for finite spectra by induction on the number of cells using the 5-lemma, and for arbitrary X by passing to direct limits. □

E-Adams Resolution

We now define the E_*-Adams resolution for a spectrum X. As the main theorem of this section, we need to reformulate it to fit our new setting.

Definition 15. An E_*-Adams resolution for X is a diagram

$$\begin{array}{ccccccc} \cdots \longrightarrow & X_2 & \xrightarrow{i_1} & X_1 & \xrightarrow{i_0} & X_0 & = X \\ & \downarrow{\scriptstyle j_2} & & \downarrow{\scriptstyle j_1} & & \downarrow{\scriptstyle j_0} & \\ & K_2 & & K_1 & & K_0 & \end{array}$$

such that for all $s \geq 0$, the following conditions hold:

(*i*) X_{s+1} is the fiber of j_s.

(*ii*) $E \otimes X_s$ is a retract of $E \otimes K_s$, meaning there exists a map $h_s \colon E \otimes K_s \to E \otimes X_s$ such that $h_s \circ (E \otimes j_s) = \mathrm{id}$. In particular, $(j_s)_* \colon E_*(X_s) \to E_*(K_s)$ is a monomorphism.

(*iii*) K_s is a retract of $E \otimes K_s$. In particular, $\pi_*(K_s) \to E_*(K_s)$ is a monomorphism.

(*iv*) $\mathrm{Ext}^{t,u}_{E_*(E)}(E_*(\mathbb{S}), E_*(K_s)) = \begin{cases} \pi_u(K_s) & \text{if } t = 0 \\ 0 & \text{else} \end{cases}$

The first three conditions are in there to mirror the definiton of the standard Adams resolution. Notably, while in the earlier setting, we required j_s to induce a surjective map on cohomology, here we instead require a monomorphism on homology, reflecting our dualized setting. More specifically the first conditions allows for the construction of the spectral sequence as befor. The second and third condition will help to prove naturaility of the spectral sequence. The fourth condition ensures that the resulting resolution of $E_*(X)$ serves to identify the E_2-page as before, by guaranteeing that $E_*(K_s)$ forms a projective resolution.

Lemma 16. *For a spectrum* X*, an* E_**-Adams resolution exists.*

Proof. We proceed inductively. As before, assume that the resolution has been constructed up to X_s. Then, set $K_s = E \otimes X_s$ and let X_{s+1} be the fiber of $\eta \otimes X_s \colon X_s = \mathbb{S} \otimes X_s \to E \otimes X_s = K_s$. Since E is a ring spectrum, it is a retract of $E \otimes E$, and thus $E \otimes X_s$ is a retract of $E \otimes K_s = E \otimes E \otimes X_s$. Being an E-module spectrum, $E \otimes K_s$ is also a retract of E. Finally, we have $E_*(K_s) = E_*(E) \otimes_{\pi_*(E)} E_*(X_s)$, and the desired form of Ext follows from the previous discussions. □

Definition 17. The E_*-Adams resolution constructed in the proof above is called the canonical E_*-Adams resolution.

Funfact. *The E_1-page of the Adams spectral sequence associated with the canonical E_*-Adams resolution is the cobar complex $C^*(E_*(X))$.*

Construction

With all of this groundwork, we can now prove Theorem 11. This proof will proceed quickly, as we have set up everything to allow for a repetition of the arguments from the previous section in an analogous way.

We begin by constructing an exact couple from the Adams resolution, as before. Following the approach of Section 1, this leads to a spectral sequence with differentials of degree $(r, r-1)$ and expected abutment $F_*^s = \mathrm{im}(i^s \colon \pi_*(X_s) \to \pi_*(X))$.

We can identify the E_2-page with a similar strategy. The long exact sequences in E-homology associated with the fibrations $X_{s+1} \to X_s \to K_s$ reduce to short exact sequences, as $(j_s)_*$ is a monomorphism by the definition of an E_*-Adams resolution. These short exact sequences splice together to form a long exact sequence:

$$0 \to E_*(X) \to E_*(K_0) \to E_*(\Sigma K_1) \to \cdots$$

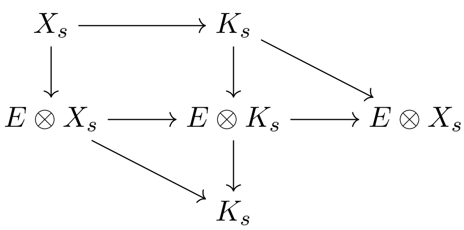
The fourth condition in the definition of E_*-Adams resolutions implies that $E_1^{s,t} = \pi_{t-s}(K_s) = \mathrm{Ext}^{0,-s}_{E_*(E)}(E_*(\mathbb{S}), E_*(K_s)) = \mathrm{Ext}^0_{E_*(E)}(E_*(\mathbb{S}), E_*(\Sigma^s K_s))$. Therefore, the E_1-page takes the form

$$\mathrm{Ext}^0_{E_*(E)}(E_*(\mathbb{S}), E_*(K_0)) \to \mathrm{Ext}^0_{E_*(E)}(E_*(\mathbb{S}), E_*(\Sigma K_1)) \to \cdots$$

Consequently, the E_2-page is the cohomology of this complex, which corresponds to the desired Ext-term, as discussed previously [1, A1.2.4].

Furthermore, we can prove conditional convergence in exactly the same way as before. When we replace X with $X_E^\wedge$, this actually results in full convergence, as we can take an E_*-Adams resolution satisfying $\varprojlim X_s = *$. Upon establishing naturality, this replacement is valid because $X \to X_E^\wedge$ induces an isomorphism in E-homology.

Naturality can be established as follows. We construct a map of spectral realizations inductively. Let $\{X_s\}$ be an arbitrary E_*-resolution of X and let R_0 be the canonical one. Define $R^n = \{X_s^n\}$ where $X_s^n = X_s$ and $K_s^n = K_s$ for $s < n$, while $K_s^n = E \otimes X_s^n$ for $s \geq n$. Then R^∞ represents the arbitrary resolution, and we construct an equivalence between R^0 and R^∞ by establishing equivalences between R^n and R^{n+1} for all n. To do this, it suffices to construct maps between K_s and $E \otimes X_s$ compatible with the map from X_s. By definition, K_s and $E \otimes X_s$ are both retracts of $E \otimes K_s$, yielding a commutative diagram:



in which the horizontal and vertical compositions are identities. It follows that the diagonal maps are the ones we seek.

To conclude, we restate the Filtration Theorem in this general setting. The proof follows in a similar manner as before.

Theorem 18 (Filtration Theorem)**.** *An element $f \in \pi_*(X)$ has Adams filtration $\geq s$ if and only if the map f can be factored into s maps, each of which becomes trivial after smashing the target with E.*

Nilpotent E-Completion

Here, I will briefly present some essential facts about $X_E^\wedge$, the nilpotent E-completion of X. First, I will provide a definition of nilpotent E-completion that is particularly useful in our context. Then, I will state a proposition that demonstrates how, in most cases, we can find a suitable model for $X_E^\wedge$, although we will omit the proof.

Definition 19. An E-completion $X_E^\wedge$ of X is a spectrum equipped with a map $X \to X_E^\wedge$ that induces an isomorphism in E-homology and ensures that $X_E^\wedge$ has an E_*-Adams resolution $\{X_s\}$ with $\varprojlim X_s = *$.

This definition becomes particularly useful when we recall the conditional convergence theorem. In fact, it is precisely this reason that allows us to converge to the homotopy groups of the nilpotent E-completion of X rather than merely to the homotopy groups of X, since for a general X, such a resolution does not always exist. This consideration also motivated our earlier replacement of X with $X_p^\wedge$. Finally, we will outline models for $X_E^\wedge$ that are relevant to our needs.

Proposition 20. *If X is a connective spectrum and E is an Adams-type ring spectrum then an nilpotent E-completion of X is given by*

$$X_E^\wedge = \begin{cases} X\mathbb{Q} & \text{if } \pi_0(E) = \mathbb{Q} \\ X_{(p)} & \text{if } \pi_0(E) = \mathbb{Z}_{(p)} \\ X & \text{if } \pi_0(E) = \mathbb{Z} \\ X_p^\wedge & \text{if } \pi_0(E) = \mathbb{F}_p \text{ and } \pi_n(X) \text{ is finitely generated for all } n \end{cases}$$

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[1] Douglas C Ravenel. *Complex cobordism and stable homotopy groups of spheres*, volume 347. American Mathematical Society, 2023.

[2] John Rognes. The adams spectral sequence. *www.mn.uio.no/math/personer/vit/rognes/papers/notes.050612.pdf*, 2012.