Bordism with involutions and an Adams spectral sequence

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Background on the Adams spectral sequence

In stable homotopy theory, one of the most powerful tools for computing stable homotopy groups of spheres $\pi_* S^0$ one prime at a time is the **Adams spectral sequence**. This can be thought of as a descent spectral sequence along the map $S^0 \to \mathbb{F}_2$. This has several nice properties that are fundamental to homotopy theory.

1. The Adams spectral sequence has an interpretation in terms of **formal groups** of infinite height. Indeed, $\mathbb{F}_2$ classifies additive formal groups, so the stack over which we take the descent spectral sequence is equivalently the moduli stack of automorphisms of the **universal additive formal group**.
2. The Adams spectral sequence can be categorified into the category of $\mathbb{F}_2$**-synthetic spectra**, a stable ∞-category of Adams resolutions. This is given by
$$\mathrm{Syn}_{\mathbb{F}_2} = \mathrm{Shv}_\Sigma^{\mathrm{Sp}}(\mathrm{Sp}_{\mathbb{F}_2}^{\mathrm{fp}})$$
and interpolates between Sp and a derived category of $(\mathbb{F}_2, \mathcal{A}_*)$-comodules (descent data).
3. There is an **Adams vanishing line** of slope 1/2 on the E_2-page for certain comodules satisfying a technical condition.
4. The category of $\mathbb{F}_2$-synthetic spectra and the Adams vanishing line can be used to build **Toda obstruction theory**: this is a recipe that starts with something algebraic (a comodule over the dual Steenrod algebra) and tells you whether you can build a spectrum with this homology.

Background on bordism with involutions

Consider an elementary abelian 2-group A. One can define a cohomology theory on A-spaces called **equivariant bordism** in terms of equivariant cobordisms of A-manifolds. This is represented by an A-spectrum denoted mO_A.

- This theory does **not** have equivariant Thom isomorphisms. We correct this by localising mO_A away from the potential Thom classes to obtain a well behaved spectrum MO_A
- Both resulting theories depend **uniformly** on the group A. Indeed, the notion of equivariant bordism and Thom classes has a certain functoriality in the group of equivariance. We say that these extend to el_2**-global spectra**.
- This global functoriality gives us a family of **short exact sequences** on homotopy groups, in particular at $A = C_2$ we get
$$0 \to \pi_\star^{C_2}\mathrm{MO}_{C_2} \xrightarrow{a_\sigma} \pi_\star^{C_2}\mathrm{MO}_{C_2} \xrightarrow{\mathrm{res}_e^{C_2}} \pi_\star^{e}\mathrm{MO}_{C_2} \to 0 \quad (1)$$
associated to the character $\sigma\colon C_2 \xrightarrow{\mathrm{id}} C_2 \cong O(1)$ that classifies the real sign representation σ. The class $a_\sigma\colon S^0 \to S^\sigma$ is the inclusion of the fixed points of the one-point-compactification.
- By a theorem of Hausmann ([Hau22]), this is part of the structure of a **global group law**: an algebraic structure that gives rise to equivariant global group laws at every group. In fact, MO_A for varying A gives the example of the **universal** 2-torsion global group law.

Bringing the two together

At the trivial group, $\pi_\star^e \mathrm{MO}_{C_2}$ recovers classical unoriented bordism $\mathrm{MO}_\star = \mathrm{MO}_*[u^\pm]$, and a classical theorem of Quillen ([Qui07]) tells us that this theory carries the universal 2-torsion formal group law. In fact, any 2-torsion formal group is isomorphic to an additive formal group (albeit not uniquely), so we obtain an isomorphism from the E_2-page onward between the the descent spectral sequences along $S^0 \to \mathbb{F}_2$ and $S^0 \to \mathrm{MO}$.

- This tells us that MO_A is a good generalisation of $\mathbb{F}_2$ to the A-equivariant setting, where A is an elementary abelian 2-group.
- Our goal is to get a good handle on the Adams spectral sequence based on MO_{C_2}, i.e. the descent spectral sequence along $S^0_{C_2} \to \mathrm{MO}_{C_2}$.

The short exact sequence in (1) gives us a Bockstein spectral sequence of the form
$$\mathrm{Ext}_{\mathrm{MO}_\star\mathrm{MO}}(\mathrm{MO}_\star, \mathrm{MO}_\star)[a_\sigma] \implies \mathrm{Ext}_{\pi_\star^{C_2}(\mathrm{MO}_{C_2}\otimes \mathrm{MO}_{C_2})}(\pi_\star^{C_2}\mathrm{MO}, \pi_\star^{C_2}\mathrm{MO}).$$
The E_1-page above, by Quillen's result, is isomorphic to
$$\mathrm{Ext}_{\mathrm{MO}_\star\mathrm{MO}}(\mathrm{MO}_\star, \mathrm{MO}_\star)[a_\sigma] \cong \mathrm{Ext}_{\mathcal{A}_*}(\mathbb{F}_2, \mathbb{F}_2)[u^\pm][a_\sigma],$$
i.e. the classical Adams spectral sequence E_2-page.

Properties of the C_2-equivariant Asseq

1. Thanks to Hausmann's theorem and (a real version of) work of Cole–Greenlees–Kriz ([CGK02]) corrected and improved in forthcoming work of Groenjes, we see that this descent spectral sequence has an interpretation in terms of C_2-equivariant 2-torsion formal groups. In particular, we can identify the descent Hopf algebroid as
$$\pi_\star^{C_2}(\mathrm{MO}_{C_2}\otimes \mathrm{MO}_{C_2}) \cong \pi_\star^{C_2}\mathrm{MO}_{C_2}[b_1^{\mathcal{F}}, b_2^{\mathcal{F}}, \ldots][(t + b_1^{\mathcal{F}} a_\sigma)^{-1}]$$
after choosing an appropriate flag $\mathcal{F}$ of a complete real C_2-universe.
2. The ring spectrum MO_{C_2} is sufficiently well behaved that one can set up a stable ∞-category of MO_{C_2}-Adams resolutions given by
$$\mathrm{Syn}_{\mathrm{MO}_{C_2}} = \mathrm{Shv}_\Sigma^{\mathrm{Sp}}(\mathrm{Sp}_{\mathrm{MO}_{C_2}}^{C_2,\mathrm{fp}})$$
interpolating between Sp^{C_2} and a derived category of $(\pi_\star^{C_2}\mathrm{MO}_{C_2}, \pi_\star^{C_2}(\mathrm{MO}_{C_2}\otimes \mathrm{MO}_{C_2}))$-comodules (descent data).
3. From the Bockstein spectral sequence, we obtain an Adams vanishing line of slope 1/2 (suitably interpreted) on the (now trigraded) E_2-page.
4. The formal setup is sufficiently general that one can also use this to construct a Toda obstruction theory for C_2-spectra in terms of their MO_{C_2}-homology.

Goals

Having an Adams spectral sequence in the equivariant world that is amenable to direct computation can be quite useful for several applications (work in progress).

- One can run the Bockstein spectral sequence to obtain classes in the equivariant E_2-page. In particular, many "classical" nonequivariant classes lift to the equivariant setting.
- Classically, the Adams spectral sequence can be computed completely for some simple comodules such as $\mathcal{A}_* /\!\!/ \mathcal{A}(1)_*$ or $\mathcal{A}_* /\!\!/ \mathcal{E}(n)_*$ corresponding to spectra of chromatic interest. What are the equivariant analogues of these?
- The equivariant Adams spectral sequence can be used to compute C_2-equivariant stable stems. These are of interest also for non-equivariant computations, since they contain information about the Mahowald invariant ([GI20]).
- In particular, can we recover the classes in the computation of the C_2-equivariant $K(1)$-local sphere due to Balderrama ([Bal21])?
- Nonequivariant Toda obstruction classes live in the E_1-page of the Bockstein spectral sequence. For certain well behaved comodules, there should be a connection between these classical obstruction classes and the equivariant Toda obstruction classes living in the equivariant E_2-page.
- Nonequivariantly, Toda obstruction theory can be used to prove the existence of certain finite spectra admitting v_2-self maps ([BE20]). Can this be done equivariantly?
- A classical construction of Priddy ([Pri80]) shows that BP can be constructed by attaching cells to the 2-local sphere spectrum to kill all odd degree classes. This cell attachment procedure and universal property of BP is detected on $\mathbb{F}_2$-homology. Can we perform a similar construction to obtain the equivariant Brown–Peterson spectrum of Wisdom ([Wis24]) and May ([May98])? In particular, this would shed light on the appropriate notion of evenness in the equivariant setting.

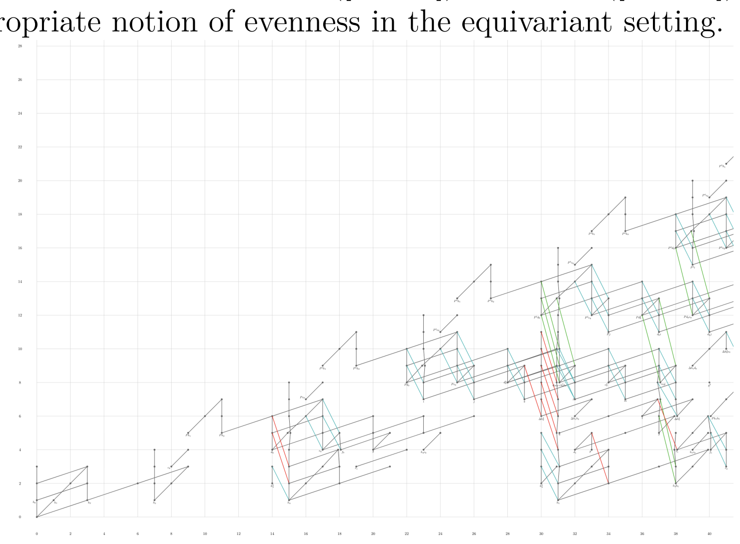

Figure 1: The classical $\mathbb{F}_2$-Adams E_2-page for the sphere. Source: ([Isa14])

Comparison with other techniques

There is an Eilenberg–MacLane functor from C_2-Mackey functors (with values in abelian groups) to C_2-spectra. In particular, one could view the constant Mackey functor $\underline{\mathbb{F}_2}$ as an appropriate equivariant generalisation of $\mathbb{F}_2$ (i.e. Bredon homology). This has been used very extensively in the equivariant chromatic literature. However, we work with MO_{C_2} because it is more directly suited to our applications.

- Bredon homology is not as well behaved as MO_{C_2}-homology: it does not have equivariant Thom isomorphisms. In particular, the coefficients $\pi_\star^{C_2}\underline{\mathbb{F}_2}$ are quite complicated (there is a negative cone), and its cooperations (descent data) are complicated as well.
- It is not clear that this is sufficiently well behaved to categorify the Adams spectral sequence based on it.
- It does not have an obvious universal property in terms of equivariant formal groups.

References

[Bal21] William Balderrama. The c_2-equivariant $k(1)$-local sphere. *arXiv preprint arXiv:2103.13895*, 2021.

[BE20] Prasit Bhattacharya and Philip Egger. A class of 2-local finite spectra which admit a v21-self-map. *Advances in Mathematics*, 360:106895, 2020.

[CGK02] Michael Cole, John PC Greenlees, and Igor Kriz. The universality of equivariant complex bordism. *Mathematische Zeitschrift*, 239:455–475, 2002.

[GI20] Bertrand J Guillou and Daniel C Isaksen. The bredon-landweber region in c_2-equivariant stable homotopy groups. *Documenta mathematica*, 25, 2020.

[Hau22] Markus Hausmann. Global group laws and equivariant bordism rings. *Annals of Mathematics*, 195(3):841–910, 2022.

[Isa14] Daniel C Isaksen. Classical and motivic adams charts. 2014.

[May98] J Peter May. Equivariant and nonequivariant module spectra. *Journal of Pure and Applied Algebra*, 127(1):83–97, 1998.

[Pri80] Stewart Priddy. A cellular construction of BP and other irreducible spectra. *Mathematische Zeitschrift*, 173:29–34, 1980.

[Qui07] Daniel Quillen. On the formal group laws of unoriented and complex cobordism theory. In *Topological Library: Part 1: Cobordisms and Their Applications*, pages 285–291. World Scientific, 2007.

[Wis24] Noah Wisdom. Properties and examples of a-landweber exact spectra. *arXiv preprint arXiv:2401.12227*, 2024.

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