BONN INTERNATIONAL GRADUATE SCHOOL OF MATHEMATICS

A motivic approach to equivariant synthetic spectra

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Background on Synthetic spectra

In stable homotopy theory, an important computational and conceptual tool is given by the **Adams–Novikov spectral sequence**, i.e. the descent spectral sequence along the unit map

$$\mathbb{1} \to \mathrm{MU}$$

which gives rise to a strongly convergent spectral sequence

$$E_2^{*,*} \cong \mathrm{Ext}^{*,*}_{\mathrm{MU}_*\mathrm{MU}}(\mathrm{MU}_*, \mathrm{MU}_* X) \implies \pi_* X$$

for any bounded below spectrum X. Unfortunately, spectral sequences are very rigid objects and it is therefore desirable to have a homotopical enhancement of this notion, i.e. a category of *formal Adams–Novikov resolutions* in which we can do homotopy theory. Such a construction can be made ([Pst23b][Pst23a]) giving rise to the category of **synthetic spectra**

$$\mathrm{Syn} = \mathrm{Fun}^{\oplus}_{\mathrm{cof}}(\mathrm{Pure}(C_1)^{\mathrm{op}}, \mathrm{Sp}),$$

where $\mathrm{Pure}(C_1)$ denotes the closure under retracts and extensions of the subcategory of Sp spanned by spheres $S^{2n}, n \in \mathbb{Z}$. Note that this actually does not make reference to MU at all!

Background on motivic spectra

Many fundamental invariants of smooth schemes over a base scheme B satisfy the following properties:

1. descent for the Nisnevich topology,
2. $\mathbb{A}^1$-homotopy invariance: their value on X is the same as on $\mathbb{A}^1 \times X$,
3. projective bundle formulas that relate their value on $\mathbb{P}^1 \times X$ and X.

These are therefore naturally represented in the category of **motivic spectra** over B defined as

$$\mathrm{SH}(B) = \mathrm{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathrm{Sm}_B; \mathrm{Ani})_*[(\mathbb{P}^1)^{\otimes -1}].$$

There exists an equivariant analogue of this construction ([Hoy17]); if G is a sufficiently nice group scheme we may consider the site of smooth B-schemes with G-actions and construction

$$\mathrm{SH}^G(B) = \mathrm{Shv}_{\mathrm{Nis},\mathbb{A}^1}(\mathrm{Sm}^G_B; \mathrm{Ani})_*[(\mathrm{Th}(\mathcal{E}))^{\otimes -1} \mid \mathcal{E} \in \mathrm{Vect}^G_B]$$

which one can equivalently think of as an extension of SH from schemes to sufficiently nice stacks including $[B/G]$. Note that $\mathrm{SH}(B)$ naturally has a bigraded family of spheres $S^{t,w} = \Sigma^{t-w}\mathbb{G}_m^w$.

Motivic and synthetic spectra

In the nonequivariant setting, there is an equivalence ([Pst23b])

$$\mathrm{Syn}_p \simeq \mathrm{SH}(\mathbb{C})^{\mathrm{cell}}_p$$

for any prime p, where the cellular category is the one generated by bigraded spheres. This allows us to connect computations in the motivic stable homotopy category over $\mathbb{C}$ to the Adams–Novikov spectral sequence and is an essential tool in computations of stable stems. The essential ingredients are the following three results.

1. Let $\mathrm{Pure}(\mu_1)$ denote the full subcategory on retracts of extensions of spheres $S^{2n,n} = (\mathbb{P}^1)^{\otimes n}$, then one can identify
$$\mathrm{SH}(\mathbb{C})^{\mathrm{cell}} \simeq \mathrm{Fun}^{\oplus}_{\mathrm{cof}}(\mathrm{Pure}(\mu_1)^{\mathrm{op}}, \mathrm{Sp})$$
as the full subcategory on functors satisfying the same condition as in the definition of Syn.
2. The key to proving this is that there exists a **motivic lift** MGL of MU such that it also admits a cell structure with cells in $\mathrm{Pure}(\mu_1)$ and furthermore
$$\pi_{2w,w}\mathrm{MGL} \cong \pi_{2w}\mathrm{MU}, \qquad \pi_{2w-k,w}\mathrm{MGL} = 0$$
for all $k > 0$.
3. For an arbitrary prime p, the functor $\mathrm{SH}(\mathbb{C}) \to \mathrm{Sp}$ induced by $X \mapsto X(\mathbb{C})^{\mathrm{an}}$ induces an equivalence
$$\mathrm{Pure}(\mu_1)/p \xrightarrow{\sim} \mathrm{Pure}(C_1)/p.$$

Equivariant algebraic cobordism

We construct $\mathrm{MGL}_G \in \mathrm{SH}^G(\mathbb{C})$ as the universal Thom spectrum, i.e. the colimit the Thom spectra of all rank zero equivariant virtual bundles over all smooth proper $\mathbb{C}$-schemes with G-action (cf. [BH21]):

$$\mathrm{MGL}_G \simeq \varinjlim_{\alpha \in K^\circ(X)} \mathrm{Th}_X(\alpha) \simeq \varinjlim_{d,i} \Sigma^{-2d,-d}\mathrm{Th}_{\mathrm{Gr}_d(V_i)}(Q^i_d).$$

- This satisfies a sort of pre-global functoriality in the group: if $f\colon [\mathbb{C}/H] \to [\mathbb{C}/G]$ is a map of quotient stacks, there exists a map
$$\alpha_f\colon f^*\mathrm{MGL}_G \to \mathrm{MGL}_H$$
which is an equivalence if f is representable (e.g. corresponds to a subgroup inclusion), this is enough to make the collection
$$\mathrm{Lattices} \cong \mathrm{Tori}^{\mathrm{op}} \to \mathrm{CRing}, T^n \mapsto \pi^{T^n}_{*,*}\mathrm{MGL}_{T^n}$$
into a **global group law** in the sense of [Hau22].
- If G is embeddable, say μ_n or GL_1, then we can construct cell structures on the Graßmannians –and therefore on MGL_G– using only cells of the form $\mathrm{Th}(V)$ for V a vector bundle on $[\mathbb{C}/G]$ (i.e. a complex G-representation).
- The Graßmannian model endows this global group law with strong regularity properties which allow us to conclude
$$\pi^{\mu_n}_{2w,w}\mathrm{MGL}_{\mu_n} \cong \pi^{C_n}_{2w}\mathrm{MU}_{C_n}, \qquad \pi^{\mu_n}_{2w-k,w}\mathrm{MGL}_{\mu_n} = 0$$
for all $k < 0$.

Equivariant synthetic reconstruction (joint with Keita Allen)

In the equivariant world, the correct replacement for the notion of having an even cell structure is that of having a cell structure by **complex** representation spheres. We therefore define $\mathrm{Pure}(C_n) \subset \mathrm{Sp}^{C_n}$ to be the full subcategory on retracts of extensions of objects of the form $S^V \otimes (C_n/C_d)_+$ with V a complex representation, and set

$$\mathrm{Syn}^{C_n} = \mathrm{Fun}^{\oplus}_{\mathrm{cof}}(\mathrm{Pure}(C_n)^{\mathrm{op}}, \mathrm{Sp})$$

as usual. We can easily deduce the following.

- Syn^{C_n} admits a **t-structure** $\tau_{c\geq\star}$ coming from Sp giving rise to a filtered spectrum
$$\mathrm{fil}^{\mathrm{ev}}_\star\mathbb{1}_{C_n} = \Gamma(\mathbb{1}_{C_n}; \tau_{c\geq\star}\mathrm{map}(-, \mathbb{1}_{C_n})).$$
- More generally, one can evaluate at other orbits and representation spheres to obtain a functor
$$\mathrm{fil}^{\mathrm{ev}}_\star\colon \mathrm{Sp}^{C_n} \to (\mathrm{Sp}^{C_n})^{\mathrm{RU}(C_n)^{\leq}}, X \mapsto \{\Gamma((C_n/C_d)_+, \tau_{c\geq V}\mathrm{map}(-, X))\}_{d|n}$$
which recovers the **equivariant Adams–Novikov filtration**.

On the motivic side, motivated by the cell structure on MGL_{μ_n}, we define $\mathrm{Pure}(\mu_n)$ to be the full subcategory of $\mathrm{SH}^{\mu_n}(\mathbb{C})$ on retracts of extensions of objects of the form $\mathrm{Th}(V) \otimes (\mu_n/\mu_d)_+$ and $\mathrm{SH}^{\mu_n}(\mathbb{C})^{\mathrm{cell}}$ to be the subcategory generated under colimits by these.

1. The vanishing result in MGL_{μ_n} formally gives us an equivalence
$$\mathrm{Mod}(\mathrm{SH}^{\mu_n}(\mathbb{C})^{\mathrm{cell}}; \mathrm{MGL}_{\mu_n}) \simeq \mathrm{Fun}^{\oplus}(\mathrm{Pure}(\mu_n)^{\mathrm{op}} \otimes \mathrm{MGL}_{\mu_n}, \mathrm{Sp})$$
cf. the **weight structure** on MGL-motives ([BKWX22], [Bon10]).
2. The cell structure on MGL_{μ_n} allows us to descend this to an identification
$$\mathrm{SH}^{\mu_n}(\mathbb{C})^{\mathrm{cell}} \simeq \mathrm{Fun}^{\oplus}_{\mathrm{cof}}(\mathrm{Pure}(\mu_n)^{\mathrm{op}}, \mathrm{Sp})$$
with the full subcategory on functors satisfying the same condition as in the definition of Syn, cf. the **heart structure** of [HP23].
3. Using **isotropy separation** and the t-structure obtained from the expression above, one can prove that taking complex points induces an equivalence
$$\mathrm{Pure}(\mu_n)/p \xrightarrow{\sim} \mathrm{Pure}(C_n)/p$$
4. Using our computation of $\pi^{\mu_n}_{2\star,\star}\mathrm{MGL}_{\mu_n}$ we may deduce from this that both categories of perfect pure objects as well as their classes of cofibre sequences are equivalent mod p, so there is an equivalence
$$\mathrm{Syn}^{C_n}_p \simeq \mathrm{SH}^{\mu_n}(\mathbb{C})^{\mathrm{cell}}_p$$

Why do we care?

- As in the nonequivariant setting, categorifying the equivariant Adams–Novikov spectral has many **computational** applications.
- This reveals a deep connection between the notion of **evenness** and algebrogeometric phenomena over $\mathbb{C}$: the torsion information in the pure Tate spheres in (equivariant) motivic spectra over $\mathbb{C}$ is determined by the complex spheres in (equivariant) spectra.
- The even filtration, which in particular generalises the motivic filtrations of [BMS19] and [BL22], naturally lives in Syn. Cyclotomic spectra can be fruitfully ([DHRY25]) described in terms of objects of $\mathrm{Sp}^{C_{p^\infty}}$, and their **equivariant even filtration** naturally lives in $\mathrm{Syn}^{C_{p^\infty}}$.
- Equivariant motivic spectra admit a notion of **equivariant norms** ([Bac22]), hence so do equivariant synthetic spectra under this equivalence. The Adams–Novikov filtration on the equivariant sphere spectrum can therefore be equipped with a highly structured equivariant multiplicative refinement.
- At more interesting groups, similar equivalences give rise to chromatic refinements of the **derived geometric Satake equivalence** ([Dev23]).

References

[Bac22] Tom Bachmann. Motivic spectral mackey functors. *arXiv preprint arXiv:2205.13926*, 2022.

[BH21] Tom Bachmann and Marc Hoyois. Norms in motivic homotopy theory. *Astérisque*, 425, 2021.

[BKWX22] Tom Bachmann, Hana Jia Kong, Guozhen Wang, and Zhouli Xu. The chow t-structure on the ∞-category of motivic spectra. *Annals of Mathematics*, 195(2):707–773, 2022.

[BL22] Bhargav Bhatt and Jacob Lurie. Absolute prismatic cohomology. *arXiv preprint arXiv:2201.06120*, 2022.

[BMS19] Bhargav Bhatt, Matthew Morrow, and Peter Scholze. Topological hochschild homology and integral p-adic hodge theory. *Publications mathématiques de l'IHÉS*, 129(1):199–310, 2019.

[Bon10] Mikhail V Bondarko. Weight structures vs. t-structures; weight filtrations, spectral sequences, and complexes (for motives and in general). *Journal of K-theory*, 6(3):387–504, 2010.

[Dev23] Sanath K Devalapurkar. Chromatic aberrations of geometric satake over the regular locus. *arXiv preprint arXiv:2303.09432*, 2023.

[DHRY25] Sanath Devalapurkar, Jeremy Hahn, Arpon Raksit, and Allen Yuan. Prismatization of commutative ring spectra (*forthcoming*), 2025.

[Hau22] Markus Hausmann. Global group laws and equivariant bordism rings. *Annals of Mathematics*, 195(3):841–910, 2022.

[Hoy17] Marc Hoyois. The six operations in equivariant motivic homotopy theory. *Advances in Mathematics*, 305:197–279, 2017.

[HP23] Peter J Haine and Piotr Pstrągowski. Spectral weight filtrations. *arXiv preprint arXiv:2309.15072*, 2023.

[HRW22] Jeremy Hahn, Arpon Raksit, and Dylan Wilson. A motivic filtration on the topological cyclic homology of commutative ring spectra. *arXiv preprint arXiv:2206.11208*, 2022.

[Pst23a] Piotr Pstrągowski. Perfect even modules and the even filtration. *arXiv preprint arXiv:2304.04685*, 2023.

[Pst23b] Piotr Pstrągowski. Synthetic spectra and the cellular motivic category. *Inventiones mathematicae*, 232(2):553–681, 2023.

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