Oberseminar: Motivic Spectra

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Contents

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1 Projective and weighted homotopy invariance

1.1 Spectrum objects

Let $\mathscr{C}$ be a presentably symmetric monoidal ∞-category, i.e. an object of $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$. Let T be an object of $\mathscr{C}$. In this section, we discuss how to define T-spectra in $\mathscr{C}$.

Definition 1.1. Let $\mathrm{SSeq}(\mathscr{C}) := \mathrm{Fun}(\mathrm{Fin}^{\simeq}, \mathscr{C})$ be the category of symmetric sequences in $\mathscr{C}$.

Informally, a symmetric sequence X in $\mathscr{C}$ is given by a sequence of objects $X(n)$ for all $n \geq 0$ with an action of Σ_n on X_n. Let us denote the initial object in $\mathscr{C}$ by $\varnothing$. Now define a symmetric sequence

$$\mathrm{sh}(T) = (\varnothing, T, \varnothing, \varnothing, \ldots) \in \mathrm{SSeq}(\mathscr{C})$$

using the informal description of symmetric sequences above. Now define

$$T^{\otimes} := \mathrm{Free}_{\mathbb{E}_\infty}(T) \simeq (\mathbb{1}, T, T^{\otimes 2}, T^{\otimes 3}, \ldots)$$

the free $\mathbb{E}_\infty$-algebra on T using the symmetric monoidal structure on $\mathrm{SSeq}(\mathscr{C})$.

Remark 1.2. Since $T^{\otimes}$ is a symmetric sequence, it automatically encodes the actions of Σ_n on $T^{\otimes n}$. This makes for better bookkeeping. Compare the free $\mathbb{E}_\infty$-algebra on the object T viewed as an object of $\mathrm{Fun}(\mathbb{N}, \mathscr{C})$; this would have n-th object $T^{\otimes n}/\Sigma_n$.

Definition 1.3. Define the category of lax T-spectra in $\mathscr{C}$ by

$$\mathrm{Sp}_T^{\mathrm{lax}}(\mathscr{C}) := \mathrm{Mod}(\mathrm{SSeq}(\mathscr{C}), T^{\otimes}).$$

Define a full subcategory $\mathrm{Sp}_T(\mathscr{C}) \subset \mathrm{Sp}_T^{\mathrm{lax}}(\mathscr{C})$ on objects $\{E_n\}_{n\geq 0}$ such that for all m the adjoint map $E_m \to E_{m+1}^T$ is an equivalence.

To unwrap the second part of the definition, let us note that an object of $\mathrm{Sp}_T^{\mathrm{lax}}(\mathscr{C})$ can be described as a symmetric sequence $\{E_n\}_{n\geq 0}$ in $\mathscr{C}$, such that its $T^{\otimes}$-module structure is encoded by the datum of $\Sigma_n \times \Sigma_m$-equivariant maps

$$T^{\otimes n} \otimes E_m \to E_{n+m}$$

for all $n, m \geq 0$. Then the map in the definition is just the adjoint of the Σ_m-equivariant map $T \otimes E_m \to E_{m+1}$ under the tensor-hom adjunction of $\mathscr{C}$.

Proposition 1.4. *The inclusion $\mathrm{Sp}_T(\mathscr{C}) \subset \mathrm{Sp}_T^{\mathrm{lax}}(\mathscr{C})$ of this full subcataegory is part of a Bousfield localisation at the class of maps*

$$\{\mathrm{sh}(T) \otimes E \to E\}_E.$$

Further, this is a symmetric monoidal localisation, so that $\mathrm{Sp}_T(\mathscr{C}) \in \mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$.

Definition 1.5. We can then define the adjunction

$$\mathscr{C} \underset{\Omega^\infty}{\overset{\Sigma_T^\infty}{\rightleftarrows}} \mathrm{Sp}_T(\mathscr{C}),$$

where Ω_T^∞ is given by evaluation at $\varnothing \in \mathrm{Fin}^{\simeq}$, and Σ_T^∞ is a symmetric monoidal functor.

Let us now make precise in what sense $\mathrm{Sp}_T(\mathscr{C})$ tensor-inverts the object T in $\mathscr{C}$.

Proposition 1.6. *Given $\mathscr{D} \in \mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$, the square*

$$\begin{tikzcd}
\mathrm{Fun}^{\otimes,\mathrm{L}}(\mathrm{Sp}_T(\mathscr{C}), \mathscr{D}) \arrow[r, "(\Sigma_T^\infty)^*"] \arrow[d] & \mathrm{Fun}^{\otimes,\mathrm{L}}(\mathscr{C}, \mathscr{D}) \arrow[d, "\mathrm{ev}_T"] \\
\mathrm{Pic}(\mathscr{D}) \arrow[r] & \mathscr{D}
\end{tikzcd}$$

is a pullback square.

Remark 1.7. In the bottom left corner, we have the Picard anima of $\mathscr{D}$. Note that this only has equivalences so the bottom horizontal arrow is a non-full inclusion (and in fact so is the top horizontal one).

Proof. Let $\mathscr{M}$ be a $\mathscr{C}$-algebra in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$. Then

1. T acts invertibly on $\mathrm{Sp}_T(\mathscr{M})$,
2. if T acts invertibly on $\mathscr{M}$, then $\Sigma_T^\infty \colon \mathscr{M} \to \mathrm{Sp}_T(\mathscr{M})$ is an equivalence

Let us now prove these two results.

1. Let $\{E_i\}_{i\geq 0}$ be an object of $\mathrm{Sp}_T(\mathscr{M})$. Note that $\mathrm{sh}(T) \simeq \mathrm{sh}(\mathbb{1}) \otimes T$, so that by assumption the map
$$E \to (E^{\mathrm{sh}(\mathbb{1})})^T \simeq (E^T)^{\mathrm{sh}(\mathbb{1})}$$
is an equivalence. In fact, we see that $(-)^T$ and $(-)^{\mathrm{sh}(\mathbb{1})}$ are inverse functors, and we conclude that the left adjoint $T \otimes -$ of $(-)^T$ is then an equivalence as well.
2. It is clear that the functors $\mathrm{ev}_{\underline{n}} \colon \mathrm{Sp}_T(\mathscr{M}) \to \mathscr{M}$ are jointly conservative. Since T acts invertibly on M by assumption, we have
$$\begin{aligned}\mathrm{ev}_{\underline{n}}(E) &\simeq \mathrm{ev}_{\underline{n-1}} E^{\mathrm{sh}(\mathbb{1})},\\ &\simeq T \otimes \mathrm{ev}_{\underline{n-1}}(E),\end{aligned}$$
so we see that equivalently just $\mathrm{ev}_\varnothing = \Omega_T^\infty$ is conservative. Therefore, its left adjoint Σ_T^∞ must be fully faithful. It remains to check whether $\Omega_T^\infty \circ \Sigma_T^\infty$ is homotopic to the identity. This is immediate for a lax T-spectrum. But since T acted invertibly, we see that the lax T-spectrum
$$\Sigma_T^\infty X = (X \otimes T^{\otimes n})_{n\geq 0}$$
is in fact already lies in the full subcategory of T-spectra, so that Σ_T^∞ is essentially surjective.

□

1.2 Construction of motivic spectra

For the rest of this lecture series, let S be a derived scheme, and denote by Sm_S the category of smooth S-schemes.

Definition 1.8. Let $F \colon \mathrm{Sm}_S^{\mathrm{op}} \to \mathscr{C}$ be a functor.

- We say that F satisfies smooth blowup excision (sbe) if for any closed immersion $Z \to X$ in Sm_S, the Cartesian square
$$\begin{tikzcd} E \arrow[r] \arrow[d] & \mathrm{Bl}_Z X \arrow[d] \\ Z \arrow[r] & X \end{tikzcd}$$
obtained by blowing up X at Z gets sent to a Cartesian square by F.
- We say that F satisfies elementary blowup excision (ebe) if the above holds whenever $Z \to X$ is of the form $Y \xrightarrow{s_0} \mathbb{A}^n_Y$, i.e. the zero section in an affine space over some $Y \in \mathrm{Sm}_S$.

Remark 1.9. Note that in the second item, the blowup square is now of the form

$$\begin{tikzcd} \mathbb{P}^{n-1}_X \arrow[r] \arrow[d] & \mathrm{Bl}_0 \mathbb{A}^n_X = \mathbb{V}_{\mathbb{P}^{n-1}_X}(\mathscr{O}(1)) \arrow[d] \\ X \arrow[r, "s_0"] & \mathbb{A}^n_X. \end{tikzcd}$$

Both horizontal maps are $\mathbb{A}^1$-equivalences so we see that $\mathbb{A}^1$-invariance implies elementary blowup excision.

Proposition 1.10. *Suppose $\mathscr{C}$ is stable and let $F\colon \mathrm{Sm}_S^{\mathrm{op}} \to \mathscr{C}$ be a functor.*

1. *If F satisfies ebe and Nisnevich descent, it satisfies sbe.*
2. *F satisfies ebe if and only if the maps $F(K_n \times X \to X)$ are equivalences for all $n \geq 0$, where we define $K_n = \mathbb{A}^n \sqcup_{\mathrm{Bl}_0 \mathbb{A}^n} \mathbb{P}^{n-1}$.*

Proof. We sketch a proof of part 1. Note that we assume Nisnevich descent so we can assume that S and X are affine. We then have a sort of tubular neighbourhood theorem, i.e. a Cartesian diagram

$$\begin{tikzcd} Z \arrow[d] \arrow[dr, phantom, "\llcorner"] & Z \arrow[l, "\mathrm{id}"'] \arrow[r, "\mathrm{id}"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & Z \arrow[d] \\ X & V \arrow[l] \arrow[r] & \mathbb{V}(\mathscr{N}_{Z/X}) \end{tikzcd}$$

where all vertical maps are closed immersions and both bottom horizontal maps are étale. □

Definition 1.11. Define the category of motivic spectra over S by

$$\mathrm{MS}_S \coloneqq \mathrm{Sp}_{\mathbb{P}^1}\mathscr{P}^{\mathrm{Sp}}(\mathrm{Sm}_S)_{\mathrm{Nis,ebe}}.$$

Define its full subcategory of $\mathbb{A}^1$-invariant motivic spectra by

$$\mathrm{MS}_S^{\mathbb{A}^1} \coloneqq \mathrm{Sp}_{\mathbb{P}^1}\mathscr{P}^{\mathrm{Sp}}(\mathrm{Sm}_S)_{\mathrm{Nis},\mathbb{A}^1}.$$

Remark 1.12. The proposition just above tells us that we have an equivalence

$$\mathrm{MS}_S \simeq \mathrm{Sp}_{\mathbb{P}^1}\mathscr{P}^{\mathrm{Sp}}(\mathrm{Sm}_S)_{\mathrm{Nis,sbe}},$$

and an equivalence

$$\mathrm{MS}_S^{\mathbb{A}^1} \simeq \mathrm{Sp}_{\mathbb{P}^1}\mathscr{P}^{\mathrm{Sp}}(\mathrm{Sm}_S)_{\mathrm{Nis},\mathbb{A}^1,\mathrm{ebe}},$$

so that this really is a full subcategory of the latter. It is clear per construction that

$$\mathrm{MS}_S^{\mathbb{A}^1} \simeq \mathrm{SH}(S)$$

recovers the Morel–Voevodsky motivic category.

1.3 Projective and weighted homotopy invariance

Let us recall the notations

$$\mathbb{V}(\mathscr{E}) \coloneqq \operatorname{Spec}\operatorname{Sym}\mathscr{E}, \qquad \mathbb{P}(\mathscr{E}) \coloneqq \operatorname{Proj}\operatorname{Sym}\mathscr{E},$$

for $\mathscr{E} \in \mathrm{QCoh}(X)$. We further denote by

$$\mathrm{Vect}(X) \subset \mathrm{Perf}(X) \subset \mathrm{QCoh}(X)$$

the full subcategory on finite locally free sheaves, and we use the same notation to refer to the left Kan extensions of these functors to $\mathscr{P}(\mathrm{Sm}_S)$.

Theorem 1.13 (Projective homotopy invariance)**.** *Let $X \in \mathscr{P}(\mathrm{Sm}_S)$, $\mathscr{E} \in \mathrm{Vect}(X)$. Suppose we are given a map $\sigma\colon \mathscr{E} \to \mathscr{O}_X$ inducing a section $\sigma\colon X \to \mathbb{V}(\mathscr{E})$. Then there is a canonical homotopy $h(\sigma)$ in $(\mathrm{MS}_S)_{/\Sigma^\infty_{\mathbb{P}_1} X_+}$ between the composites*

$$X \underset{0}{\overset{\sigma}{\rightrightarrows}} \mathbb{V}(\mathscr{E}) \hookrightarrow \mathbb{P}(\mathscr{E} \oplus \mathscr{O}).$$

Let us begin by considering a few simple consequences that tell us how to think of this theorem as a homotopy invariance result.

Definition 1.14. Given a category $\mathscr{C}$ tensored over $\mathrm{Sm}_{\mathbb{Z}}$, and maps $f, g : X \to Y$ in $\mathscr{C}$,

- a $\mathbb{P}^1$-homotopy $f \simeq_{\mathbb{P}^1} g$ is a map
$$h\colon X \otimes \mathbb{P}^1 \to Y$$
such that h restricts to f and g on $X \times 0$ and $X \times 1$ respectively.
- A weight n $\mathbb{A}^1$-homotopy $f \simeq_{\mathbb{A}^1/^n\mathbb{G}_m} g$ for some integer n is a map
$$h\colon X \otimes \mathbb{A}^1 \to Y$$
which is $\mathbb{G}_m$-equivariant and restricts to f resp. g on $X \times 0$ resp. $X \times 1$. Here the $\mathbb{G}_m$-action on both X and Y is trivial, but $\mathbb{G}_m$ acts on $\mathbb{A}^1$ with weight n, i.e. by putting the element t in $\mathbb{A}^1 \simeq \mathrm{Spec}(\mathbb{Z}[t])$ in graded degree n.

Remark 1.15. From the definition above, we see that a weight 0 $\mathbb{A}^1$-homotopy is just a classical $\mathbb{A}^1$-homotopy.

Corollary 1.16. *In* MS_S

1. *$\mathbb{P}^1$-homotopic maps are homotopic.*
2. *Weight n $\mathbb{A}^1$-homotopic maps are homotopic if $n \neq 0$.*

Proof. Both are simple applications of the theorem above

1. Set $\mathscr{E} = \mathscr{O}$, $\sigma = \mathrm{id}$.
2. Assume $n \geq 1$, then there is a map $\mathbb{P}^1 \to \mathbb{A}^1/\mathbb{G}_m$ classifying the Cartier divisor $n \cdot \{0\}$ on $\mathbb{P}^1$. This map clearly[1] sends 0 to 0 and 1 to 1 so it suffices to precompose the $\mathbb{A}^1/\mathbb{G}_m$-homotopy with this map to obtain a $\mathbb{P}^1$-homotopy, then apply the first part of the corollary. When $n \leq -1$, we note that $\mathbb{A}^1/^n\mathbb{G}_m \simeq \mathbb{A}^1/^{-n}\mathbb{G}_m$, i.e. the sign of the action can be flipped up to equivalence so we are back in the case $n \geq 1$.

□

Let us now prove projective homotopy invariance in the simpler case where $\mathscr{E}$ is of rank one. The more general case is still true but the proof just becomes harder. Furthermore, the rank one case is all we explicitly need in forthcoming applications.

Proof. Letting $\sigma\colon \mathscr{E} \to \mathscr{O}$ be the map given in the statement of the theorem, define an element
$$e_{21}(\sigma) \coloneqq \begin{pmatrix} \mathrm{id}_{\mathscr{E}} & 0 \\ \sigma & \mathrm{id}_{\mathscr{O}} \end{pmatrix} \in \mathrm{Aut}(\mathscr{E} \oplus \mathscr{O}).$$

It is clear that this gives an automorphism e of $\mathbb{P}(\mathscr{E} \oplus \mathscr{O})$ such that it acts as the identity on the subspace $\mathbb{P}(\mathscr{E})$. Therefore, we obtain an induced action of e on $\mathbb{P}(\mathscr{E} \oplus \mathscr{O})/\mathbb{P}(\mathscr{E}) = \mathrm{Th}_X(\mathscr{E})$. Let us write $\overline{e}$ for this corresponding automorphism of the Thom space. Note that the following two diagram commute, the right one per construction of $e_{21}(\sigma)$, and the left one by taking projective bundles of the right hand side.

$$\begin{tikzcd} & \mathbb{P}(\mathscr{E} \oplus \mathscr{O}) \arrow[dd, "e"] \\ X \arrow[ur, "0"] \arrow[dr, "\sigma"'] & \\ & \mathbb{P}(\mathscr{E} \oplus \mathscr{O}) \end{tikzcd} \qquad \begin{tikzcd} & \mathscr{E} \oplus \mathscr{O} \arrow[dl, "\pi_2"'] \\ \mathscr{O} & \\ & \mathscr{E} \oplus \mathscr{O}. \arrow[ul, "\sigma+\mathrm{id}"] \arrow[uu, "e_{21}(\sigma)"'] \end{tikzcd}$$

[1] This is where we exclude the case $n = 0$, since in that case the map would classify the trivial Cartier divisor, hence factor through 0.

Therefore, it suffices to show that $e \simeq \mathrm{id}$ in $(\mathrm{MS}_S)/X$. We will begin by proving the weaker claim that $\overline{e} \simeq \mathrm{id}$ in MS_S. This will be proven by increasing the size of this matrix from 2 to 3. Note that all constructions are sufficiently functorial and compatible with base change that we can assume $X = S$ and forget about the slice in the statement of the proof.

To prove that $\overline{e}$ is homotopic to the identity, begin by observing that $\mathrm{Th}_X\mathscr{E}$ is a $\otimes$-invertible object in MS_S. Indeed, this is obvious when the rank of $\mathscr{E}$ is one, as then locally on S $\mathscr{E}$ is trivial and we have $\mathrm{Th}_S\mathscr{E} \simeq \mathbb{P}^1$. Furthermore, note that we can construct a diagram

$$\begin{tikzcd}
\mathbb{P}(\mathscr{E} \oplus \mathscr{E} \oplus \mathscr{O}) & & \mathbb{P}(\mathscr{E} \oplus \mathscr{O}) \times \mathbb{P}(\mathscr{E} \oplus \mathscr{O}) \\
& B \arrow[ul] \arrow[ur] & \\
& \partial B \arrow[u] \arrow[dl] \arrow[dr] & \\
\mathbb{P}(\mathscr{E} \oplus \mathscr{E}) \arrow[uuu] & & (\mathbb{P}(\mathscr{E}) \times \mathbb{P}(\mathscr{E} \oplus \mathscr{O})) \cup (\mathbb{P}(\mathscr{E} \oplus \mathscr{O}) \times \mathbb{P}(\mathscr{E})) \arrow[uuu]
\end{tikzcd}$$

where B is either of the equivalent spaces obtained as the blowup of $\mathbb{P}(\mathscr{E} \oplus \mathscr{O})$ in the two points $\mathbb{P}(\mathscr{E}) \sqcup \mathbb{P}(\mathscr{E})$ or the blowup of $\mathbb{P}(\mathscr{E} \oplus \mathscr{O}) \times \mathbb{P}(\mathscr{E} \oplus \mathscr{O})$ at the point $\mathbb{P}(\mathscr{E}) \times \mathbb{P}(\mathscr{E})$. We see that the exceptional divisor in this blowup, denoted ∂B maps to the subspaces as in the diagram making it commute. The point is that one can then take cofibres of all the vertical maps to obtain a span

$$\begin{tikzcd}
\mathrm{Th}_S(\mathscr{E} \oplus \mathscr{E}) \arrow[loop left, "e_{31}(\sigma)"] & B/\partial B \arrow[l] \arrow[r] & \mathrm{Th}_S(\mathscr{E})^{\wedge 2}. \arrow[loop right, "\overline{e} \wedge \mathrm{id}"]
\end{tikzcd}$$

Now note that all maps in this span are equivalences by smooth blowup excision in MS_S, and in fact compatibly with the endomorphisms above. It therefore suffices to prove that $e_{31}(\sigma)$ is homotopic to the identity, where

$$e_{31}(\sigma) = \begin{pmatrix} \mathrm{id}_{\mathscr{E}} & 0 & 0 \\ 0 & \mathrm{id}_{\mathscr{E}} & 0 \\ \sigma & 0 & \mathrm{id}_{\mathscr{O}} \end{pmatrix}.$$

But note by the Steinberg relations for elementary matrices that we have a commutator expression

$$e_{31}(\sigma) = [e_{32}(\sigma), e_{21}(\mathrm{id}_{\mathscr{E}})] \in \mathrm{Aut}\mathrm{Th}_S(\mathscr{E} \oplus \mathscr{E}).$$

However, we noted that $\mathrm{Th}_S(\mathscr{E})$ was $\otimes$-invertible, while smooth blowup excision tells us that $\mathrm{Th}_S(\mathscr{E} \oplus \mathscr{E}) \simeq \mathrm{Th}_S(\mathscr{E})^{\wedge 2}$, so that the latter is $\otimes$-invertible as well. This means that its automorphism spectrum admits a canonical lift from an $\mathbb{E}_1$-group to an $\mathbb{E}_\infty$-group, whence this commutator vanishes and we obtain $e_{31}(\sigma) \simeq \mathrm{id}$. This proves the weaker claim.

To prove the stronger claim, let us still assume $\mathscr{E}$ is of rank one, then we can consider the defining fibre sequence in MS_S of the form

$$\begin{tikzcd}
\mathbb{P}(\mathscr{E})_+ \arrow[loop left, "\mathrm{id}"] \arrow[r] & \mathbb{P}(\mathscr{E} \oplus \mathscr{O})_+ \arrow[loop right, "e"] \arrow[d] \\
& \mathrm{Th}_S(\mathscr{E}) \arrow[loop right, "\overline{e}"]
\end{tikzcd}$$

Note that this is split by the the zero section of $\mathbb{P}(\mathscr{E} \oplus \mathscr{O})$ so we obtain a splitting

$$\begin{tikzcd} \mathbb{P}(\mathscr{E} \oplus \mathscr{O})_+ \arrow[r, "\sim"] \arrow[loop left, "e"] & \mathbb{1}_S \oplus \mathrm{Th}_S(\mathscr{E}) \arrow[loop right, "\mathrm{id}\oplus\overline{e}"] \end{tikzcd}$$

□

Remark 1.17. Note that we have used $\mathbb{P}^1$-stability to prove projective homotopy invariance, which is why we define MS_S as a category of $\mathbb{P}^1$-spectrum objects from the start. Contrast this to SH, where $\mathbb{A}^1$-invariance already happens unstable (well, it is in the definition).

Example 1.18. Consider two vector bundles $\mathscr{E}$, $\mathscr{F}$ on S, then the inclusions

$$\mathbb{P}(\mathscr{E}) \to \mathbb{P}(\mathscr{E} \oplus \mathscr{F}) \leftarrow \mathbb{P}(\mathscr{F})$$

are disjoint and complementary up to $\mathbb{A}^1$-homotopy. In MS_S we can say something similar, namely that the diagram

$$\begin{tikzcd} \mathbb{P}(\mathscr{E} \oplus \mathscr{F}) \setminus \mathbb{P}(\mathscr{F}) \arrow[d, "\pi"] \arrow[dr, hook, "\circ" marking] & \\ \mathbb{P}(\mathscr{E}) \arrow[r, hook, "/" marking] & \mathbb{P}(\mathscr{E} \oplus \mathscr{F}) \end{tikzcd}$$

commutes using a twisted $\mathbb{P}^1$-homotopy. This is given by

$$h\colon \mathbb{P}_{\mathbb{P}(\mathscr{E}\oplus\mathscr{F})\setminus\mathbb{P}(\mathscr{F})}(\mathscr{F}(-1) \oplus \mathscr{O}) \to \mathbb{P}(\mathscr{E} \oplus \mathscr{F})$$

which we can define on points. A point of the source is given by a surjection $\phi\colon \mathscr{E} \oplus \mathscr{F} \twoheadrightarrow \mathscr{L}$ onto a line bundle such that its restriction to $\mathscr{E}$ is still surjective, as well as a further surjection $\mathscr{F} \otimes \mathscr{L}^{-1} \oplus \mathscr{O} \twoheadrightarrow \mathscr{M}$ onto a further line bundle. We send this to the point in the target given by

$$\mathscr{E} \oplus \mathscr{F} \xrightarrow{\phi|_{\mathscr{E}}\oplus\mathrm{id}} \mathscr{L} \oplus \mathscr{F} \simeq (\mathscr{F} \otimes \mathscr{L}^{-1} \oplus \mathscr{O}) \otimes \mathscr{L} \twoheadrightarrow \mathscr{M} \otimes \mathscr{L}.$$

Corollary 1.19. *Any two linear embeddings $\mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^n$ are homotopic in MS_S.*

Corollary 1.20. *Setting $\mathscr{E} = \mathscr{O}, \mathscr{F} = \mathscr{O}^n$ above, we see that $\mathbb{A}^n \hookrightarrow \mathbb{P}^n$ is homotopic to the constant map at zero in MS_S.*

Remark 1.21. The slogan of projective homotopy invariance is then: *$\mathbb{A}^1$-homotopic arguments still kind of work as long as you restrict to projective things; and this at the cost of having to work stably.*

2 The Bass fundamental theorem

The classical Bass delooping construction in K-theory is closely related to the motive of $\mathbb{G}_m$. Note that this is $\otimes$-invertible in $\mathrm{SH}(S)$, but this is no longer the case in MS_S, because we can no longer do the usual trick of writing its suspension as $\mathbb{P}^1$. More precisely, let us consider the adjunctions

$$\begin{tikzcd} \mathrm{MS}_S = \mathrm{Sp}_{\mathbb{P}^1}\mathscr{P}(\mathrm{Sm}_S;\mathrm{Sp})_{\mathrm{Nis,ebe}} \arrow[r, shift left, "\Omega^\infty"] & \mathrm{Sp}_{\mathbb{P}^1}\mathscr{P}(\mathrm{Sm}_S;\mathrm{Ani}_*)_{\mathrm{Nis,ebe}} = \mathrm{MS}_S^{\mathrm{un}} \arrow[l, shift left, "\Sigma^\infty"] \\ \mathrm{SH}(S) \arrow[u, hook] \arrow[r, shift left, "\Omega^\infty"] & \mathrm{Sp}_{\mathbb{P}^1}\mathscr{P}(\mathrm{Sm}_S;\mathrm{Ani}_*)_{\mathrm{Nis},\mathbb{A}^1}. \arrow[u, hook] \arrow[l, shift left, "\Sigma^\infty"] \end{tikzcd}$$

Note that the bottom adjunction is an equivalence by what we said before: there is a commutative diagram

$$\begin{tikzcd} \mathbb{G}_m \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathbb{A}^1 \arrow[r] \arrow[d] & * \arrow[dd] \\ \mathbb{A}^1 \arrow[r] \arrow[d] & \mathbb{P}^1 \arrow[dr, dotted, "\partial"] & \\ * \arrow[rr] & & S^1 \otimes \mathbb{G}_m \end{tikzcd}$$

and it suffices to note that the maps $\mathbb{A}^1 \to *$ are equivalences in $\mathrm{Sp}_{\mathbb{P}^1}\mathscr{P}(\mathrm{Sm}_S;\mathrm{Ani}_*)_{\mathrm{Nis},\mathbb{A}^1}$ to see that ∂ is an equivalence in this category, so that S^1 is already $\otimes$-invertible in this category and it is stable, whence its stabilisation adjunction is an equivalence. The comparison map ∂ still exists in the non-$\mathbb{A}^1$-invariant setting, it will simply not be an equivalence, but still plays an important role.

Theorem 2.1 (Bass fundamental theorem). *In MS_S, there is a decomposition*

$$\mathbb{G}_m \simeq \Sigma^{-1}\mathbb{P}^1 \oplus \mathbb{A}^1 \oplus (\mathbb{P}^1 \setminus 0),$$

so that ∂ is not an equivalence but an inclusion of the first summand.

Theorem 2.2 (Bass delooping). *The functor*

$$\Omega^\infty \colon \mathrm{MS}_S \to \mathrm{MS}_S^{\mathrm{un}}$$

is fully faithful and has essential image given by those objects $E \in \mathrm{MS}_S^{\mathrm{un}}$ such that the morphism $\mathrm{id}_E \otimes \partial \colon E \otimes \mathbb{P}^1 \to E \otimes \Sigma\mathbb{G}_m$ admits a retraction.

Remark 2.3. To further contrast this with the $\mathbb{A}^1$-invariant setting, let us remark that $\mathbb{G}_m$ is not even a dualisable object in MS_S. On the other hand, $\mathrm{SH}(S)$ should be rigid assuming resolution of singularities over S

Example 2.4. We give a proof of this non-dualisability by way of example. Note that de Rham cohomology is representable in MS_S by an algebra object dR, and therefore has a Künneth formula, so that its value on a dualisable object should once again be dualisable. However, a classical computation tells us that $\mathrm{dR}_{\mathbb{G}_m}$ is infinite-dimensional.

Let us now prove the Bass fundamental theorem.

Proof. Consider the Mayer–Vietoris sequence coming from the pushout above, which is of the form

$$\mathbb{G}_m \to \mathbb{A}^1 \oplus \mathbb{P}^1 \setminus 0 \to \mathbb{P}^1 \xrightarrow{\partial} \Sigma\mathbb{G}_m \to \Sigma\mathbb{A}^1 \oplus \Sigma\mathbb{P}^1 \setminus 0 \to \cdots.$$

We want to show that this final three-term section depicted above is split, for which it suffices to find a retraction of ∂. Note that this is the same datum as nullhomotopies of the maps

$$\mathbb{A}^1 \hookrightarrow \mathbb{P}^1, \qquad\qquad \mathbb{P}^1 \setminus 0 \hookrightarrow \mathbb{P}^1,$$

where all spaces here are pointed at 1. Now recall from Corollary 1.20 that the map $\mathbb{A}^1_+ \to \mathbb{P}^1_+$ factors through $*_+$. Since we are working in the stable setting we have $\mathbb{A}^1_+ \simeq \mathbb{A}^1 \oplus \mathbb{1}$ and $\mathbb{P}^1_+ \simeq \mathbb{P}^1 \oplus \mathbb{1}$, so we can just restrict to the first factor every time to see that the map $\mathbb{A}^1 \to \mathbb{P}^1$ is nullhomotopic. The same argument goes through for $\mathbb{P}^1 \setminus 0$. □

2.1 Proof of Bass delooping

The proof of Bass delooping goes through many steps, so we write it as a separate section. By Zariski descent, it is safe to assume that the base S is qcqs, whence MS_S and $\mathrm{MS}_S^{\mathrm{un}}$ are compactly generated by motives of smooth S-schemes.

Definition 2.5. An element $E \in \mathrm{MS}_S^{\mathrm{un}}$ is called fundamental if $\mathrm{id}_E \otimes \partial$ admits a retract. These span a full subcategory $\mathrm{MS}_S^{\mathrm{fd}}$.

Recall that we want to show, in particular, that the essential image of Ω^∞ is $\mathrm{MS}_S^{\mathrm{fd}}$.

Remark 2.6. One inclusion is obvious. Indeed, by the Bass fundamental theorem, we see that ∂ is a summand inclusion in MS_S, whence anything in the image of Ω^∞ is fundamental just by taking the image of the retraction of this summand inclusion by Ω^∞.

Remark 2.7. Note that E being fundamental can equally well be phrased in terms of E^∂ admitting a section.

Proof. A retraction would be a map living in

$$\begin{aligned}\mathrm{map}(E \otimes \Sigma\mathbb{G}_m, E \otimes \mathbb{P}^1) &\simeq \mathrm{map}((\mathbb{P}^1)^{-1} \otimes E \otimes \Sigma\mathbb{G}_m, E),\\ &\simeq \mathrm{map}(E^{\mathbb{P}^1}, E^{\Sigma\mathbb{G}_m}).\end{aligned}$$

The middle equivalence holds since $\mathbb{P}^1$ is tensor-invertible, and we see that a retraction in the first mapping anima precisely corresponds to a section in the last mapping anima. □

Remark 2.8. It is truly remarkable that $\mathrm{MS}_S^{\mathrm{fd}}$ turns out to be the essential image of MS_S under a fully faithful right adjoint (hence equivalent to MS_S). Indeed, the condition of a certain map admitting a retract is not at all clearly closed under any sort of categorical operations! We see a posteriori, that this retraction in $\mathrm{MS}_S^{\mathrm{fd}} \subset \mathrm{MS}_S^{\mathrm{un}}$ will even be unique, since there is clearly a canonical choice in MS_S.

Definition 2.9. For $E, X \in \mathrm{MS}_S^{\mathrm{un}}$, define the E-cohomology of X by

$$E^{p,q}(X) := \pi_{2q-p}\mathrm{map}(X, \Sigma^q_{\mathbb{P}^1} E), 2q - p \geq 0.$$

We also adopt the notational convention

$$E^n(X) := E^{2n,n}(X) = \pi_0\mathrm{map}(X, \Sigma^n_{\mathbb{P}^1} E).$$

Remark 2.10. Since $\mathrm{MS}_S^{\mathrm{un}}$ is not stable, these mapping animæ are just that–so that these cohomology objects are not necessarily abelian groups, and are a priori not well defined for $2q - p < 0$. However, note that invertibility of $\mathbb{P}^1$ gives us a suspension isomorphism

$$E^{p,q}(X) \cong E^{p+2,q+1}(\mathbb{P}^1 \otimes X)$$

Remark 2.11. If E is fundamental, we can use the retraction to obtain a split short exact sequence

$$0 \to E^{p+1,q+1}(\mathbb{A}^1 \otimes X)^{\oplus 2} \to E^{p+1,q+1}(\mathbb{G}_m \otimes X) \xrightarrow{\partial^*} E^{p,q}(X) \to 0 \tag{1}$$

of (abelian) groups or pointed sets. Now note that if $2q - p = 0$ resp. $2q - p = 1$, we see that $2(q+1) - (p+1) = 2$ resp. $2(q+1) - (p+1) = 1$ so that the first two terms are (abelian) groups and the cofibre term uniquely admits this structure as well. We conclude that if $2q - p \geq 0$ and E is fundamental, $E^{p,q}(X)$ uniquely admits the structure of an abelian group.

This observation tells us that the homotopy category $\mathrm{hMS}_S^{\mathrm{fd}}$ is an additive 1-category. Indeed, it has finite sums and products since the existence of a retraction to $\mathrm{id}_E \otimes \partial$ is clearly closed under these, and we see by this remark that it is enriched in abelian groups. This tells us that $\mathrm{MS}_S^{\mathrm{fd}}$ is an additive category. This will be used to apply Brown representability.

2.1.1 The Bass construction

For E in $\mathrm{MS}_S^{\mathrm{fd}}$, p, q two integers, we want to extend to negative E-cohomology groups $E^{p,q}(X) \in \mathrm{Ab}$ such that

1. the short exact sequence 1 is still split exact for all $p, q \in \mathbb{Z}$, and
2. there are isomorphisms $E^{p,q}(X) \cong E^{p+1,q}(\Sigma X)$ for all $p, q \in \mathbb{Z}$.

Remark 2.12. This is to be compared with Bass' construciton of negative K-groups, in which $\mathrm{K}_{-1}(X)$ is a sumand of $\mathrm{K}_0(\mathbb{G}_m \times X)$.

Let us now recall the Brown representability theorem.

Definition 2.13. Let $\mathscr{C}$ be a compactly generated pointed category, then a cohomology theory on $\mathscr{C}$ is a pair (E^*, ∂) of a functor $E^* \colon \mathscr{C}^{\mathrm{op}} \to \mathrm{Ab}^{\mathbb{N}_{\geq 0}}$ such that

- every E^n takes arbitrary coproducts to products.
- A pushout square

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & \ulcorner & \downarrow \\ C & \longrightarrow & D \end{array}$$

  induces an exact sequence

$$E^n(D) \to E^n(B) \oplus E^n(C) \twoheadrightarrow E^n(A)$$

  for all n.
- ∂ is a natural isomorphism $E^n \cong E^{n+1} \circ \Sigma$ for all n.

Theorem 2.14 (Brown representability). *Let $\mathscr{C}$ be a compactly generated pointed ∞-caqttegory, then every cohomology theory (E^*, ∂) on $\mathscr{C}$ is represented by an object E of $\mathrm{Sp}(\mathscr{C})$. More precisely, we have a commutative diagram*

$$\begin{array}{ccc} \mathrm{Sp}(\mathscr{C}) & \xrightarrow{\sim} & \varprojlim(\cdots \xrightarrow{\Omega} \mathscr{C} \xrightarrow{\Omega} \mathscr{C}) \\ \downarrow & & \downarrow \\ \mathrm{CohThy}(\mathscr{C}) & \xrightarrow{\sim} & \varprojlim(\cdots \xrightarrow{\Omega} \mathrm{h}\mathscr{C} \xrightarrow{\Omega} \mathrm{h}\mathscr{C}). \end{array}$$

To apply this to our situation, let us note that the functor Ω on $\mathrm{MS}_S^{\mathrm{un}}$ restricts to $\mathrm{MS}_S^{\mathrm{fd}}$, by our observation that fundamentality can be written as the existence of a section, which is preserved by the cotensor with S^1.

Lemma 2.15. *The endofunctor*

$$\Omega \colon \mathrm{MS}_S^{\mathrm{fd}} \to \mathrm{MS}_S\mathrm{fd}$$

is an equivalence.

Proof. Since $\mathrm{MS}_S^{\mathrm{fd}}$ is additive, it suffices to show this is an equivalence on the level of homotopy categories[2]. Let us construct this inverse on the level of homotopy categories by means of Brown representability Note that every $E \in \mathrm{MS}_S^{\mathrm{fd}}$ defines a cohomology theory on $\mathrm{MS}_S^{\mathrm{un}}$ given by $(E^{*,0}, \partial)$ with suspension isomorphisms baked into the construction. Therefore, we can represent it by an object of $\mathrm{Sp}(\mathrm{MS}_S^{\mathrm{un}})$. Modelling the latter as an Ω-spectrum, we see that the functor

$$E^{1,0}(-)\colon \mathrm{hMS}_S^{\mathrm{un,op}} \to \mathrm{Ab}$$

is representably by some object of $\mathrm{hMS}_S^{\mathrm{un}}$ which we denote $E^{1,0}$. We claim that the construction $E \mapsto E^{1,0}$ is inverse to $E \mapsto \Omega E$. Three things need to be checked.

1. First, show that $E^{1,0}$ is once again fundamental.
2. Show that the map $\Omega E^{1,0} \to E$ is an equivalence. This part is immediate from the definition.
3. Show that the map $(\Omega E)^{1,0} \to E$ is an equivalence. This follows from an application of the Bass exact sequence to see that these represent the same functor.

Let us give an outline of how the last step goes. Consider the map of split short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & (\Omega E)^{2,1}(\mathbb{A}^1 \otimes X)^{\oplus 2} \arrow[r] \arrow[d] & (\Omega E)^{2,1}(\mathbb{G}_m \otimes X) \arrow[r] \arrow[d] & (\Omega E)^{1,0}(X) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & E^{1,1}(\mathbb{A}^1 \otimes X)^{\oplus 2} \arrow[r] & E^{1,1}(\mathbb{G}_m \otimes X) \arrow[r] & E^{0,0}(X) \arrow[r] & 0.
\end{tikzcd}$$

Note that the first two vertical maps are isomorphisms by the second point. We conclude by the five-lemma that the last map is an isomorphism as well. ◻

This then concludes our argument, since we have a chain of equivalences

$$\mathrm{MS}_S \simeq \varprojlim(\cdots \xrightarrow{\Omega} \mathrm{MS}_S^{\mathrm{un}} \xrightarrow{\Omega} \mathrm{MS}_S^{\mathrm{un}}) \simeq \varprojlim(\cdots \xrightarrow{\Omega} \mathrm{MS}_S^{\mathrm{fd}} \xrightarrow{\Omega} \mathrm{MS}_S^{\mathrm{fd}}) \simeq \mathrm{MS}_S^{\mathrm{fd}},$$

where we used that the essential image of Ω^∞ was contained in $\mathrm{MS}_S^{\mathrm{fd}}$, and in the last step that Ω is a self-equivalence of $\mathrm{MS}_S^{\mathrm{fd}}$.

2.2 K-theory as a motivic spectrum

Let us closed with an application of the Bass fundamental theorem and Bass delooping machinery.

Definition 2.16. Let $\mathscr{C} \in \mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ and suppose we are given a family of maps $A = \{\alpha_i\colon A_i \to \mathbb{1}\}_{i \in I}$ to the unit of $\mathscr{C}$. We say that an object X of $\mathscr{C}$ is A-periodic if for all $i \in I$ the map

$$\alpha_i^*\colon X \to X^{A_i}$$

is an equivalence. This defines a full subcategory $P_A\mathscr{C} \subset \mathscr{C}$.

Remark 2.17. In fact, the inclusion $P_A\mathscr{C} \subset \mathscr{C}$ admits a left adjoint making this into a symmetric monoidal Bousfield localisation.

[2] This is where additivity really comes in. The idea is that we can reduce checking on mapping animæ to checking on their homotopy groups. In a semiadditive category there are no canonical basepoints for mapping animæ so we would have to check at all basepoints. In an additive category, these are canonically based at zero, so checking equivalences on all mapping animæ can be done at the canonical basepoint, where it reduces to checking that this functor induces isomorphism on mapping abelain groups, i.e. on the level of additive homotopy categories.

Remark 2.18. Suppose that A is a singleton, given by $A = \{\alpha\colon T \to \mathbb{1}\}$. Then there is a commutative diagram with an equivalence

$$\begin{tikzcd} \mathscr{C} \arrow[r] \arrow[d] & \mathrm{Sp}_T(\mathscr{C}) \arrow[d] \\ P_A\mathscr{C} \arrow[r, "\sim"] & P_A\mathrm{Sp}_T(\mathscr{C}). \end{tikzcd}$$

This can be seen either manually by including into $\mathrm{Sp}_T^{\mathrm{lax}}(\mathscr{C})$ and seeing that A-periodic objects and spectrum objects are defined by the same condition, or in terms of their universal properties when mapping out to another commutative algebra in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ as in Proposition 1.6.

Now let P denote $P = \mathscr{P}(\mathrm{Sm}_S; \mathrm{Ani}_*)$, and consider the object $\mathscr{K} \in P$ given by the K-theory animæ of smooth S-schemes. This admits a natural $\mathbb{E}_\infty$-algebra structure, so we can consider $\mathrm{Mod}(P; \mathscr{K}) \in \mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$.

Let $\beta\colon \mathbb{P}^1 \otimes \mathscr{K} \to \mathscr{K}$ be the Bott class in $\mathscr{K}(\mathbb{P}^1)$. Then as above we say that $M \in \mathrm{Mod}(P, \mathscr{K})$ is Bott-periodic if it is periodic for the class $\{\mathbb{P}^1 \otimes \mathscr{K} \to \mathscr{K}\}$, i.e. if the map

$$\beta^*\colon M \to M^{\mathbb{P}^1}$$

of $\mathscr{K}$-modules is an equivalence. It is then classically known that the unit $\mathscr{K}$ is itself a periodic element. By the remark above, we have a symmetric monoidal equivalence

$$P_\beta\mathrm{Mod}(P; \mathscr{K}) \simeq P_\beta\mathrm{Sp}_{\mathbb{P}^1}(\mathrm{Mod}(P; \mathscr{K})),$$

and the image of $\mathscr{K}$ under this equivalence is the $\mathbb{E}_\infty$-ring denoted KGL.

Remark 2.19. Note that this Bott periodicity is a consequence of the projective bundle formula, which itself follows directly from the semiorthogonal decomposition of perfect complexes on projective space together with the fact that K-theory is a localising invariant. Therefore, we can repeat this process for any localising invariant. This will be discussed further on.

3 The moduli stack of vector bundles

The goal of today's lecture is to prove the following analogue of a well-known result of Morel–Voevodsky in $\mathbb{A}^1$-homotopy theory.

Theorem 3.1 (Grassmannian model for classifying space). *The map*

$$\mathrm{Gr}_n \to \mathrm{BGL}_n$$

from the motive of the infinite Grassmannian to the classifying space of GL_n *is an equivalence in* MS_S *for all* S.

In the special case $n = 1$, this is saying that the map $\mathbb{P}^\infty \to \mathrm{Pic}$ from infinite projective space to the moduli of line bundles is an equivalence.

Remark 3.2. The relevance of this theorem is that the right hand side has a universal property in terms of vector bundles, so that in particular its cohomology should recover a theory of Chern classes. The let hand side can be constructed explicitly, so that its cohomology can be computed rather explicitly as well.

3.1 Recollection on $\mathbb{A}^1$-invariant setup

Let us temporarily recall what happens in $\mathbb{A}^1$-homotopy theory, and what goes wrong in the non-$\mathbb{A}^1$-invariant setting. Recall that we define

- $\mathrm{Gr}_n = \mathrm{St}_n/\mathrm{GL}_n$ is the quotient of the Stiefel variety of n-frames by its free GL_n-action.
- $\mathrm{BGL}_n = */\mathrm{GL}_n$ is the homotopy orbits of the point.

Further, note that since GL_n is a nice group, both of these quotients can be constructed in Zariski sheaves. This means that we have explicit simplicial models in terms of action simplicial objects in $\mathcal{S}\mathrm{hv}_{\mathrm{Zar}}$.

$$\mathrm{Gr}_n \simeq \left|\mathrm{St}_n \times \mathrm{GL}_n^{\times\bullet}\right|,$$
$$\mathrm{BGL}_n \simeq \left|* \times \mathrm{GL}_n^{\times\bullet}\right|.$$

It therefore suffices to show that St_n is contractible in $\mathcal{S}\mathrm{hv}_{\mathrm{Zar},\mathbb{A}^1}$. This is the content of the proposition below.

Proposition 3.3. *Let* $F\colon \mathrm{Aff}^{\mathrm{op}} \to \mathrm{Ani}$ *be a functor satisfying the following.*

1. *Closed gluing:* F *takes pushouts along closed immersions to pullbacks.*
2. *Closed lifting: given* $Z_0 \to Z$ *a closed immersion, the map* $F(Z) \to F(Z_0)$ *is an epimorphism of animæ, i.e. surjective on* π_0.

Then F *is* $\mathbb{A}^1$*-contractible.*

Proof. We can write the $\mathbb{A}^1$-localisation of F explicitly as

$$L_{\mathbb{A}^1}F(-) = |F(\mathbb{A}^\bullet \times -)|.$$

Viewing the right hand side as a simplicial object, note that there are lifts

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & F(\mathbb{A}^\bullet \times -). \\ \Delta^n \arrow[ur, dotted] & \end{tikzcd}$$

Indeed for all X, the top horizontal arrow classifies a compatible family of $n+1$ points of $F(\mathbb{A}^{n-1}_X)$. Bly closed gluing this can be uniquely lifted to a point of $F(\partial A^n_X)$, and now closed lifting along $\partial\mathbb{A}^n_X \to \mathbb{A}^n_X$ gives us a lift to $F(\mathbb{A}^n_X)$, i.e. a lift in this diagram. We conclude that $F(\mathbb{A}^\bullet_X) \to *$ is a trivial Kan fibration in simplicial objects of $\mathcal{S}\mathrm{hv}_{\mathrm{Zar}}$. ◻

Now we can apply this proposition to $\mathrm{St}_n \mid_{\mathrm{Aff}^{\mathrm{op}}}$. Indeed, the first condition is true for any representable, and we note that St_n is Ind-representable. Since we restricted to affines, the second condition is just an exercise in commutative algebra. We conclude that

$$L_{\mathbb{A}^1}\mathrm{St}_n \mid_{\mathrm{Aff}^{\mathrm{op}}} \simeq *,$$

so that St_n is $\mathbb{A}^1$-contractible in $\mathcal{S}\mathrm{hv}_{\mathrm{Zar}}$.

Remark 3.4. More generally, for any linear algebraic group $G \subset \mathrm{GL}_n$ which is flat and finitely presented, we can show that for any G-torsor $T \to X$, the associated bundle $\mathrm{St}_n \times_G T \to X$ is an $\mathbb{A}^1$-equivalence on affines, so also $\mathrm{St}_n/B \to \mathrm{B}G$ is an equivalence in $\mathrm{MS}^{\mathbb{A}^1}$. Note that G might no longer be nice so we need to view these as fppf quotients in general.

In our non-$\mathbb{A}^1$-invariant setting, we will have a modified argument which holds for any $G \subset \mathrm{GL}_n$ such that

- St_n/G and $\mathrm{B}G$ can be computed as Nisnevich quotients, (this is true for special groups such as $\mathrm{Sp}_{2n}, \mathrm{SL}_n$), and
- there exists a copy of $\mathbb{G}_m$ inside G which acts with constant weight on $\mathbb{A}^n$.

Note that the second condition is extremely restrictive. It holds for GL_n by picking out diagonal matrices, but for exaple excludes SL_2 since the copy of $\mathbb{G}_m$ on matrices of the form $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ acts with weight $(1,-1)$ on $\mathbb{A}^2$. It is for this reason that we will only present the proof for $G = \mathrm{GL}_n$.

3.2 Proof of Grassmannian models

Let us first establish some notation.

Definition 3.5. For $\mathscr{E}, \mathscr{F}$ finite locally free sheaves on S, let us write

$$\mathrm{St}(\mathscr{E}, \mathscr{F})$$

for the scheme classifying surjections $\mathscr{E} \twoheadrightarrow \mathscr{F}$. This admits an open embedding into the vector bundle $\mathbb{V}(\mathscr{E}, \mathscr{F})$ classifying linear maps $\mathscr{E} \to \mathscr{F}$. Further, write St_n for $\varinjlim_k \mathrm{St}(\mathscr{O}^k, \mathscr{O}^n)$.

Remark 3.6. In stark contract to the $\mathbb{A}^1$-invariant proof, we will show that the open embedding above will be a homotopy equivalence, but the Stiefel spaces will not be contractible.

Lemma 3.7. *Let $\mathscr{E}$ be a GL_n-representation over S such that $\mathbb{G}_m$ acts with nonzero weight n. Then the map*

$$p \colon \mathbb{V}(\mathscr{E})/\mathrm{GL}_n \to \mathrm{BGL}_n$$

is an equivalence in MS_S.

Proof. Let s be the zero section of $\mathscr{E}$, then $s \circ p \simeq \mathrm{id}$ is a weight n $\mathbb{A}^1$-homotopy equivalence. Indeed, consider the map

$$\mathbb{A}^1/^n\mathbb{G}_m \times \mathbb{V}(\mathscr{E})/\mathrm{GL}_n \to \mathbb{V}(\mathscr{E})/\mathrm{GL}_n, (t, v) \mapsto (t^n v).$$

□

Theorem 3.8. *Let $\mathscr{E}, \mathscr{F}$ be finite locally free sheaves on S, and suppose there exists an epimorphism $\chi \colon \mathscr{E}^m \twoheadrightarrow \mathscr{F}$ for some $m \gg 0$. Then the open immersion*

$$\mathrm{St}_n(\mathscr{E}^\infty, \mathscr{F}) \to \mathbb{V}(\mathscr{E}^\infty, \mathscr{F})$$

is an equivalence in MS_S.

Remark 3.9. This is quite useful in general, in the case where $\mathscr{F} = \mathscr{E} = \mathscr{O}$, this tells us that the inclusion $\mathbb{A}^\infty \setminus 0 \to \mathbb{A}^\infty$ is an equivalence.

Proof. Consider the compactification $\mathbb{V}(\mathscr{E}^k, \mathscr{F}) \to \mathbb{P}(\mathscr{E}^k \otimes \mathscr{F}^\vee \oplus \mathscr{O})$, which we denote P_k. Denote its boundary by ∂P_k, and let Z_k be the closure of $\mathbb{V}(\mathscr{E}^k, \mathscr{F}) \setminus \mathrm{St}(\mathscr{E}^k, \mathscr{F})$ inside P_k. Similarly, write $\partial Z_k = Z_k \cap \partial P_k$. Now it is clear from its construction as a projective space that P_k admits an open cover by $\mathbb{V}(\mathscr{E}^k, \mathscr{F})$ and $P_k \setminus Z_k$, so we have a pushout

$$\begin{tikzcd}
\mathrm{St}(\mathscr{E}^k, \mathscr{F}) \arrow[r, hook] \arrow[d, hook] & P_k \setminus Z_k \arrow[d, hook] \\
\mathbb{V}(\mathscr{E}^k, \mathscr{F}) \arrow[r, hook] & P_k \setminus \partial Z_k.
\end{tikzcd}$$

Since MS_S is stable, this is also a pullback, so to prove the theorem it suffices to show that

$$\varprojlim_k (P_k \setminus Z_k \to P_k \setminus \partial Z_k)$$

is an equivalence. Now note that $m\mathbb{N} \to \mathbb{N}$ is a (co?)final inclusion of posets for any $m \geq 1$ (this is the m from the theorem statement), and consider the diagram

$$\begin{tikzcd}
P_k \setminus Z_k \arrow[r, hook] \arrow[d, hook] & P_{k+m} \setminus Z_{k+m} \arrow[r, hook] \arrow[d, hook] & \cdots \arrow[r, hook] & P_\infty \setminus Z_\infty \arrow[d, hook] \\
P_k \setminus \partial Z_k \arrow[r, hook] \arrow[ur, dashed, "f_k"] & P_{k+m} \setminus \partial Z_{k+m} \arrow[r, hook] \arrow[ur, dashed, "f_{k+m}"] & \cdots \arrow[r, hook] & P_\infty \setminus \partial Z_\infty. \arrow[u, dashed, bend right, "f_\infty"']
\end{tikzcd}$$

The dashed arrows are lifts we will construct for every $k + im$, $i \geq 0$ in order to obtain an inverse in the limit. Note that constructing such an f_k is the same as solving a lifting problem

$$\begin{tikzcd}
\Delta^1 \times \Delta^1 \arrow[r] \arrow[d] & \mathrm{MS}_S, \\
\Delta^3 \arrow[ur, dashed] &
\end{tikzcd}$$

where the top horizontal arrow represents each successive commutative diagram in the large diagram above. Indeed, commutativity supplies the remaining edge of Δ^3 as the down-and-right diagonal arrow along with its 2-simplices witnessing commutativity, so it just remains to construct f_k as the remaining edge of Δ^3, and fill the two faces and the 3-face adjacent to it.

Let $e_k \colon P_k \to P_{k+m}$ denoe the stabilisation map, which sends a point of the source, i.e. a surjection of the form $\mathscr{E}^k \otimes \mathscr{F}^\vee \oplus \mathscr{O} \twoheadrightarrow \mathscr{L}$ to the point of the target given by

$$\mathscr{E}^{k+m} \otimes \mathscr{F}^\vee \oplus \mathscr{O} \twoheadrightarrow \mathscr{E}^k \otimes \mathscr{F}^\vee \oplus \mathscr{O} \twoheadrightarrow \mathscr{L}$$

obtained by first projecting onto the first k factors. Define $f_k \colon P_k \to P_{k+m}$ to be the map that sends a point in the source as above to the point in the target given by

$$\begin{aligned}
\mathscr{E}^{k+m} \otimes \mathscr{F}^\vee \oplus \mathscr{O} &\xrightarrow{\chi} (\mathscr{E}^k \oplus \mathscr{F}) \otimes \mathscr{F}^\vee \oplus \mathscr{O} \\
&\xrightarrow{\sim} \mathscr{E}^k \otimes \mathscr{F}^\vee \oplus \mathscr{F} \otimes \mathscr{F}^\vee \oplus \mathscr{O} \\
&\xrightarrow{\mathrm{ev}} \mathscr{E}^k \otimes \mathscr{F}^\vee \oplus \mathscr{O} \oplus \mathscr{O} \\
&\xrightarrow{\nabla} \mathscr{E}^k \otimes \mathscr{F}^\vee \oplus \mathscr{O} \\
&\twoheadrightarrow \mathscr{L}.
\end{aligned}$$

It is easy to check that all the maps above are still surjections ($\mathscr{F}$ must have rank at least one, but this is assumed). We now claim that $e_k \simeq f_k$ by a twisted $\mathbb{P}^1$-homotopy

$$h_k \colon \mathbb{P}_{\mathbb{P}_k}(\mathscr{O}(-1) \oplus \mathscr{O}) \to P_{k+m}.$$

On points, this can be described explicitly. A point in the source is a pair (ϕ, ψ) of surjections

$$\begin{aligned} \phi &: \mathscr{E}^k \otimes \mathscr{F}^\vee \oplus \mathscr{O} \twoheadrightarrow \mathscr{L}, \\ \psi &: \mathscr{L}^\vee \oplus \mathscr{O} \twoheadrightarrow \mathscr{M}. \end{aligned}$$

We then define $h_k(\phi, \psi)$ to be the surjection

$$\begin{aligned} (\mathscr{E}^m \otimes \mathscr{F}^\vee) \oplus (\mathscr{E}^k \otimes \mathscr{F}^\vee \oplus \mathscr{O}) &\xrightarrow{\phi} (\mathscr{E}^m \otimes \mathscr{F}^\vee) \oplus \mathscr{L} \\ &\xrightarrow{\chi} (\mathscr{F} \otimes \mathscr{F}^\vee) \oplus \mathscr{L} \\ &\xrightarrow{\mathrm{ev}} \mathscr{O} \oplus \mathscr{L} \\ &\xrightarrow{\sim} (\mathscr{L}^\vee \oplus \mathscr{O}) \otimes \mathscr{L} \\ &\xrightarrow{\psi} \mathscr{M} \otimes \mathscr{L}. \end{aligned}$$

It is straightforward to check that this restricts appropriately. Indeed, $\mathbb{V}_{P_k}(\mathscr{O}(-1))$ has to canonical sections, one denoted s and given by the map $\mathscr{O}(-1) \to \mathscr{O}$, and the other given by the zero section. By projective homotopy invariance, we see that the two composites

$$P_k \underset{s}{\overset{0}{\rightrightarrows}} \mathbb{P}_{P_k}(\mathscr{O}(-1) \oplus \mathscr{O})$$

are homotopic in MS_S, but clearly $h_k \circ 0 = e_k$ while $h_k \circ s = f_k$. To finalise the proof, it then suffices to check that

- f_k restricts to a map $P_k \setminus \partial Z_k \to P_{k+m} \setminus Z_{k+m}$,
- h_k restricts to $P_k \setminus Z_k \to P_{k+m} \setminus Z_{k+m}$, and
- h_k restricts to $P_k \setminus \partial Z_k \to P_{k+m} \setminus \partial Z_{k+m}$.

This is straightforward and not particularly interesting. □

We can now bring together Lemma 3.7 and Theorem 3.8 to conclude that the composite

$$\mathrm{Gr}_n = \mathrm{St}_n/\mathrm{GL}_n \to \mathbb{V}_n/\mathrm{GL}_n \to \mathrm{BGL}_n$$

is an equivalence in MS_S.

Remark 3.10. Note that the Theorem really shows that the map $\mathrm{St}_n \to \mathbb{V}_n$ is an equivalence before taking quotients, and for this to induce an equivalence in MS_S we need these quotients to be computed in the Nisnevich topology, whence the restriction to nice groups.

3.3 Prelude to oriented motivic spectra

In MS_S we have a canonical map of pointed objects $(\mathbb{P}^1, \infty) \to (\mathrm{Pic}, *)$ classifying the line bundle $\mathscr{O}(1)$ on $\mathbb{P}^1$. By the discussion above we also know that this is homotopic to the inclusion $\mathbb{P}^1 \to \mathbb{P}^\infty \simeq \mathrm{BGL}_1 \simeq \mathrm{Pic}$

Definition 3.11. An orientation on a motivic spectrum E is a retraction of the map

$$\mathbb{P}^1 \otimes E \to \mathrm{Pic} \otimes E.$$

Remark 3.12. Note that this is different from the usual definition of orientations in homotopy theory. Indeed we do not require E to have any multiplicative structure, or for this retract to be multiplicative or E-linear. This more general setup will mean that Chern classes will be operations on E-theory and not just elements.

From this, we can already extract two constructions which hint at a theory of Chern classes.

- For any $X \in \mathscr{P}(\mathrm{Sm}_S)$ and $\mathscr{L} \in \mathrm{Pic}(X)$ classified by a map $\mathscr{L}\colon X \to \mathrm{Pic}$, we can consider the composite

$$X_+ \otimes E \xrightarrow{\mathscr{L}} \mathrm{Pic} \otimes E \to \Sigma_{\mathbb{P}_1} E,$$

  with the second map being given by the orientation. This adjoints over to a map

$$E^{X_+} \to \Sigma_{\mathbb{P}_1} E^{X_+}$$

  which we call $c_1(\mathscr{L})$, the first Chern class of $\mathscr{L}$.

- If E is a homotopy ring spectrum, we can choose the retraction to be E-linear without loss of generality, so that $c_1(\mathscr{L})$ becomes an E-linear map, i.e. a class

$$c_1 \in \widetilde{E}^{2,1}(\mathrm{Pic}).$$

  This is such that its restriction $c_1(\mathscr{O}(1)_{\mathbb{P}^1}) \in \widetilde{E}^{2,1}(\mathbb{P}^1) \cong E^{0,0}(S)$ along $\mathscr{O}(1)\colon \mathbb{P}^1 \to \mathrm{Pic}$ is the unit 1 in this ring.

Next lecture, we will see that these Chern classes are used to state the projective bundle formula theorem of Annala-Iwasa.

Theorem 3.13 (Projective bundle formula). *Let E be an oriented motivic spectrum over S, $X \in \mathscr{P}(\mathrm{Sm}_S), \mathscr{E} \in \mathrm{Vect}_r(X)$, then the map*

$$\sum_{i=0}^{r-1} c_1(\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1))^i \colon \bigoplus_{i=1}^{r-1} \Sigma_{\mathbb{P}^1}^{-i} E^{X_+} \to E^{\mathbb{P}(\mathscr{E})_+}$$

is an equivalence.

4 Oriented motivic spectra

Let us use the definition of orientations from the last lecture to define higher Chern classes and Thom isomorphisms.

Lemma 4.1. *Suppose $E \in \mathrm{CAlg}(\mathrm{hMS}_S)$ is oriented. Let X be a smooth S-scheme with an open cover $\{U_1, \dots, U_n\}$. Let $\{\mathscr{L}_i, \mathscr{L}_i'\}_{i=1}^n \subset \mathrm{Pic}(X)$ be a collection of line bundles such that for every i there is an equivalence*

$$\mathscr{L}_i \mid_{U_i} \simeq \mathscr{L}_i' \mid_{U_i} .$$

Then the product

$$\prod_{i=1}^{n} (c_1(\mathscr{L}_i) - c_1(\mathscr{L}_i')) \in \mathbb{E}^n(X)$$

vanishes.

Remark 4.2. We can draw some immediate consequences from easy cases.

- If X admits a finite trivialising cover, i.e. setting $\mathscr{L}' = \mathscr{O}$, then $c_1(\mathscr{L})$ is nilpotent in $E^*(X)$.
- In particular, using the standard cover of $\mathbb{P}^n$, we see that $c_1(\mathscr{O}_{\mathbb{P}^n}(1))^{n+1} = 0$ in $E^{n+1}(\mathbb{P}^n)$.

Proof. Let $\gamma_i = c_1(\mathscr{L}_i) - c_1(\mathscr{L}_i') \in E^1(X)$ denote the difference, so that $\gamma_i \mid_{U_i} = 0$ as a class in $E^1(U_i)$ for all i. Since this lies in the kernel of restriction to U_i, we obtain a relative class

$$\widehat{\gamma}_i \in E^1(X, U_i)$$

from the long exact sequence of the pair (X, U_i). Then we use the exterior cap product to construct the class

$$\prod_{i=1}^{n} \widehat{\gamma}_i \in E^*(X, \bigcup_{i=1}^{n} U_i).$$

Since the U_i's form a cover by assumption we identify the latter group with $E^1(X, X) = 0$. Since this product lifted the class $\prod_{i=1}^n \gamma_i$, the latter must be zero too. □

We can now prove the projective bundle formula, i.e. Theorem 3.13.

Proof. Recall that we wanted to show that the map

$$\sum_{i=0}^{r-1} c_1(\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1))^i \colon \bigoplus_{i=0}^{r-1} \Sigma_{\mathbb{P}^1}^{-i} E^{X_+} \to E^{\mathbb{P}(\mathscr{E})_+}$$

is an equivalence. We will prove the statement by induction on $r - 1 = n$. Note that by naturality and descent the statement for fixed rank is directly implied if we show it assuming $X \in \mathrm{Sm}_S$ and $\mathscr{E} = \mathscr{O}^{n+1}$ (so $r = n+1$ in the theorem statement).

$n = 0$ This case is trivial.

$n = 1$ We know essentially per definition that

$$E^{X_+} \oplus \Sigma_{\mathbb{P}^1}^{1} E^{X_+} \to E^{\mathbb{P}^1_{X+}}$$

is an equivalence, as this is just the $\mathbb{P}^1$-suspension isomorphism■.

Nik: I dont see how this is the suspetion isomorphism? Maybe you can explain me an easier way to see this than the following.

Note that the map $\mathbb{P}^1_+ \to \mathrm{Pic}$ classified by $\mathscr{O}_{\mathbb{P}^1}(1)$ factors through (pointed) $\mathbb{P}^1$. From this and the fact that further composing with the orientation gives the map $\mathbb{P}^1_+ \to \mathbb{P}^1$ identifying $+$ and ∞ one checks that the map

$$\Sigma_{\mathbb{P}^1}^{-1} E \to E^{\mathbb{P}^1}$$

induced by the first Chern class of the above line bundle identifies with the map induced by $\mathbb{P}^1_+ \to \mathbb{P}^1$.

Now the claimed isomorphism comes from the split cofiber sequence

$$*_+ \to \mathbb{P}^1_+ \to \mathbb{P}^1$$

Let us consider the blowup

$$\begin{tikzcd} D \arrow[r] \arrow[d] & Q \arrow[l, bend left, "q"] \arrow[d, "\pi"] \\ 0 \arrow[r] & \mathbb{P}^n. \end{tikzcd}$$

where $D = \mathbb{P}^{n-1}$, $Q = \mathbb{P}_D(\mathscr{O}_D(1) \oplus \mathscr{O})$ with the natural projection map $q\colon Q \to D$. We claim that $\mathscr{O}_q(1) \simeq \pi^*\mathscr{O}_{\mathbb{P}^n}(1)$. This can be checked on points. Indeed, points of $\mathbb{P}^n$ are surjections $\mathscr{O}^{n+1} \twoheadrightarrow \mathscr{L}$, while points of Q –per definition of a blowup– are factorisations

$$\begin{tikzcd} \mathscr{O}^n \arrow[r, two heads] \arrow[d, hook] & \mathscr{M} \arrow[d] \\ \mathscr{O}^{n+1} \arrow[r, two heads] & \mathscr{L}. \end{tikzcd}$$

This data is equivalent to a point in $\mathbb{P}^{n-1}$ classified by the top horizontal arrow, and a map $\mathscr{M} \oplus \mathscr{O} \twoheadrightarrow \mathscr{L}$. The latter can be seen to be precisely the points of $\mathbb{P}_{\mathbb{P}^{n-1}}(\mathscr{O}(1) \oplus \mathscr{O}) = Q$. We conclude from this description that the twisting sheaf $\mathscr{O}_q(1)$ is equivalent to $\pi^*\mathscr{O}_{\mathbb{P}^n}(1)$.
Let $s_0, s_\infty\colon D \to Q$ be the sections at zero and infinity respectively, and consider the open cover of Q given by

$$U_1 = Q \setminus s_\infty D, \qquad\qquad U_2 = Q \setminus s_0 D.$$

We see that there are canonical identifications $\mathscr{O}_q(1) \simeq \mathscr{O}$ on the former and $\mathscr{O}_q(1) \simeq q^*\mathscr{O}_{\mathbb{P}^{n-1}}(1)$ on the latter. We now apply Lemma 4.1 to see that

$$(c_1(\mathscr{O}_q(1)) - c_1(\mathscr{O})) \cdot (c_1(\mathscr{O}_Q(1)) - c_1(q^*\mathscr{O}_{\mathbb{P}^{n-1}}(1))) = 0.$$

Since we know first Chern classes are natural and are such that the first Chern class of a trivial bundle is zero, we can rewrite this as

$$c_1(\mathscr{O}_q(1)) \cdot q^*c_1(\mathscr{O}_{\mathbb{P}^{n-1}}(1)) = c_1(\mathscr{O}_q(1))^2.$$

This means that the following diagram commutes.

$$\begin{tikzcd} \bigoplus_{i=1}^n \Sigma^{-i}_{\mathbb{P}^1} E \arrow[r, "\sum_{i=1}^n c_1(\mathscr{O}(1))^i"] \arrow[d, "\sum_{i=2}^n c_1(\mathscr{O}(1))^{i-1}"'] & E^{\mathbb{P}^n/0} \arrow[d, "E^\pi"] \\ \Sigma^{-1}_{\mathbb{P}_1} E^{\mathbb{P}^{n-1}_+} \arrow[r, "c_1(\mathscr{O}_q(1)) \cdot q^*"'] & E^{Q/D}. \end{tikzcd}$$

The map E^π is an equivalence by smooth blowup excision, the left vertical map is an equivalence by the inducion hypothesis, and the bottom horizontal map is an equivalence in general. We conclude that the top horizontal map is an equivalence, whence we are done. □

Corollary 4.3. *Let $E \in \mathrm{CAlg}(\mathrm{hMS}_S)$ be oriented.*

- *We can compute*
$$E^*(\mathbb{P}^n_X) \cong E^*(X)[c]/c^{n+1},$$
where $c = c_1(\mathscr{O}_{\mathbb{P}^n_X}(1))$.

- *As n goes to infinity on the express above, we find*

$$E^*(\mathrm{Pic}\times X) \cong E^*(X)[\![c]\!],$$

where $c = c_1(\mathscr{L}^{\mathrm{univ}})$ is the first Chern class of the universal line bundle on Pic.

- *The commutative group structure on* Pic *coming from the tensor product of line bundles gives a formal group law over $E^*(S)$ when we set $X = \mathrm{Pic}$ above.*

4.1 Thom isomorphisms and higher Chern classes

Using the same notation as above, let us note that per definition there are two fibre sequences

$$\begin{tikzcd}
E^{\mathrm{Th}_X\mathscr{E}} \arrow[r] & E^{\mathbb{P}(\mathscr{E}\oplus\mathscr{O})_+} \arrow[r] & E^{\mathbb{P}(\mathscr{E})_+} \\
\Sigma^{-r}_{\mathbb{P}^1}E^{X_+} \arrow[u, dashed] & \bigoplus_{i=0}^{r}\Sigma^{-i}_{\mathbb{P}^1}E^{X_+} \arrow[l] \arrow[u] & \bigoplus_{i=0}^{r-1}\Sigma^{-i}_{\mathbb{P}^1}E^{X_+}. \arrow[l, hook'] \arrow[u]
\end{tikzcd}$$

The top one comes from the definition of $\mathrm{Th}_X\mathscr{E}$, while the bottom one is just the inclusion of a summand. The vertical arrows are the equivalences in the projective bundle formula, making the right hand side of the diagram commute. The summand inclusion on the bottom hence provides a section of the top fibre sequence, and provides the dashed equivalence called the Thom isomorphism

$$t(\mathscr{E})\colon \Sigma^{-r}_{\mathbb{P}^1}E^{X_+} \xrightarrow{\sim} E^{\mathrm{Th}_X\mathscr{E}}.$$

Remark 4.4. If E admits the structure of a homotopy ring spectrum, we are once again free to interpret this as a class $t(\mathscr{E})$ in $\widetilde{E}^r(\mathrm{Th}_X\mathscr{E})$, called the Thom class of $\mathscr{E}$.

Definition 4.5. Using the same notation as above, for $0 \le i \le r$ define a map

$$c_i(\mathscr{E})\colon \Sigma^{-i}_{\mathbb{P}^1}E^{X_+} \to E^{X_+}$$

as the unique map such that the splitting in the bottom fibre sequence above has the formula

$$\sum_{i=0}^{r}(-1)^{r-i}c_{r-i}(\mathscr{E})\Sigma^{-r}_{\mathbb{P}^1}E^{X_+} \to \bigoplus_{i=0}^{r}\Sigma^{\mathbb{P}_1-i}E^{X_+}.$$

That is, up to a sign and suspension the i-th Chern class is the map obtained from this splitting after projecting to the $(r-i)$'th factor.

Remark 4.6. Since this is a splitting, in particular we should get a nullhomotopy

$$\sum_{i=0}^{r}(-1)^{r-i}c_1(\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1))^i c_{r-i}(\mathscr{E}) \simeq 0\colon E^{X_+} \to E^{\mathbb{P}(\mathscr{E})_+}.$$

Example 4.7. As a special case, we should recover the Euler class as the top Chern class, i.e. $e(\mathscr{E}) = c_r(\mathscr{E})$. We note that the diagram

$$\begin{tikzcd}
\Sigma^{-r}_{\mathbb{P}^1}E^{X_+} \arrow[r, "e(\mathscr{E})"] \arrow[d, "t(\mathscr{E})"] & E^{X_+} \arrow[d, "\pi^*"] \arrow[dd, bend left=60, "\mathrm{id}"] \\
E^{\mathrm{Th}_X\mathscr{E}} \arrow[r] & E^{\mathbb{P}(\mathscr{E}\oplus\mathscr{O})_+} \arrow[d, "s^*"] \\
 & E^{X_+}
\end{tikzcd}$$

commutes up to the sign $(-1)^r$, so that the Euler class really is the pullback of the Thom class along any section, confirming our homotopy theoretic intuition.

4.2 Cohomology of Grassmannians

The goal of this section is to prove the following result.

Theorem 4.8. *If $E \in \mathrm{CAlg}(\mathrm{hMS}_S)$ is oriented, then*

$$E^*(X \times \mathrm{BGL}_n) E^*(X)[\![c_1, \dots, c_n]\!].$$

Recall that last time, we shoed that the map $\mathrm{Gr}_n \to \mathrm{BGL}_n$ is an equivalence in MS_S. We will instead compute the cohomology of the left hand side.

Theorem 4.9 (Grassmann bundle formula). *Let M be an E-module, hence also an oriented motivic spectrum. Let $\mathscr{E} \in \mathrm{Vect}_r(X)$, and denote by $\mathscr{Q}$ the tautological bundle on $\mathrm{Gr}_n(\mathscr{E})$. Then the map*

$$\sum_\alpha c^\alpha(\mathscr{Q}) \colon \bigoplus_\alpha \Sigma_{\mathbb{P}^1}^{-|\alpha|} M^{X_+} \to M^{\mathrm{Gr}_n(\mathscr{E})_+}$$

is an equivalence. In this notation, α ranges over multiindices $\alpha = (\alpha_1, \dots, \alpha_n)$ such that $\sum_i \alpha_i \leq r - n$ and $|\alpha| := \sum_i i\alpha_i$. We also set

$$c^\alpha := \prod_i c_i^{\alpha_i}.$$

The key to extracting this from the projective bundle formula is the Whitney sum formula for Chern classes. In the sequel, let us consider the Chern classes as packaged together in the total Chern class

$$c(\mathscr{E}) = \sum_i c_i(\mathscr{E}) t^i \in E^*(X)[t].$$

Proposition 4.10. *Let*

$$\mathscr{E}' \to \mathscr{E} \to \mathscr{E}''$$

be a short exact sequence of finite locally free sheaves on X. Then we can identify

$$c(\mathscr{E}) = c(\mathscr{E}')c(\mathscr{E}'') \in E^*(X)[t]$$

Proof. Let $\mathrm{Flag}(\mathscr{E})$ be the (derived) scheme over X parametrising full flags of $\mathscr{E}$. Then it is clear from the projective bundle formula that pulling back to this induces an injection on E-cohomology. We can therefore safely assume that $\mathscr{E}$ has a filtration

$$0 = \mathscr{E}_0 \subset \mathscr{E}_1 \subset \cdots \subset \mathscr{E}_{r-1} \subset \mathscr{E}_r = \mathscr{E}$$

of length r such that every quotient $\mathscr{E}_i/\mathscr{E}_{i-1} = \mathscr{L}_i$ is of rank one. In this case, the Whitney sum formula reduces to showing that

$$c(\mathscr{E}) = \prod_i (1 + c_1(\mathscr{L}_i)t)$$

To proceed, let us provide a splitting trick. Letting

$$\mathscr{E}' \xrightarrow{f} \mathscr{E} \to \mathscr{E}''$$

be a short exact sequence of finite locally free sheaves as above, let us consider the homogeneous coordinates t_0 and t_1 on $\mathbb{P}^1_X$, and use them to define a cofibre sequence in $\mathrm{Vect}(\mathbb{P}^1_X)$

$$\mathscr{E}' \xrightarrow{\mathrm{id}\otimes t_0 \oplus f \otimes t_1} \mathscr{E}'(1) \oplus \mathscr{E}(1) \to \widetilde{\mathscr{E}}.$$

Note that per construction

$$\begin{aligned} i_0^* \widetilde{\mathscr{E}} &\simeq \mathscr{E}' \oplus \mathscr{E}'', \\ i_\infty^* \widetilde{\mathscr{E}} &\simeq \mathscr{E}. \end{aligned}$$

Since this constitutes an open cover, we get that $c(\mathscr{E}) = c(\mathscr{E}' \oplus \mathscr{E}'')$. We are therefore free to assume now that $\mathscr{E} \simeq \bigoplus_{i=1}^r \mathscr{L}_i$ for line bundles $\mathscr{L}_i$. Consider the universal quotient $\mathscr{E} \to \mathscr{O}(1)$ on $\mathbb{P}(\mathscr{E})$, which restricts to a map $\mathscr{L}_i \to \mathscr{O}(1)$ hence a global section of $\mathscr{L}_i(1)^\vee$. Let $D_i \subset \mathbb{P}(\mathscr{E})$ be the vanishing locus of s_i. Then we see that the intersection of all D_i is empty per construction, so that $\prod_i c_1(\mathscr{L}_i(1)^\vee) = 0$. As a consequence we get

$$\xi = c_1(\mathscr{O}(1)) = c_1(\mathscr{L}_i \otimes \mathscr{L}_i^\vee) = c_1(\mathscr{L}_i) + c_1(\mathscr{L}_i(1)^\vee) + \cdots$$

up to first order by the formal group law for line bundles. We conclude that $\prod_i(\xi - c_1(\mathscr{L}_i)) = 0$ and conclude. □

We can now use this to prove the Grassmann bundle formula in Theorem 4.9 by induction. This was confusing in the lectures and it's (not really) well explained in Annala–Iwasa 2022 so we leave this to the reader…

4.3 Example: K-theory

The prime example of an oriented motivic spectrum is $\mathrm{KGL} \in \mathrm{CAlg}(\mathrm{MS}_S)$. This is a motivic spectrum such that the following hold.

- For any $X \in \mathrm{Sm}_S$, we have that
$$\mathrm{Map}(\Sigma^\infty_{\mathbb{P}_1} X_+, \mathrm{KGL}) \simeq \mathrm{K}^B(X)$$
recovers the nonconnective K-theory of X.
- KGL is β-periodic for $\beta\colon (\mathbb{P}^1, \infty) \to \mathrm{KGL}$ the Bott class given by
$$\beta = \mathscr{O}_{\mathbb{P}^1} - \mathscr{O}_{\mathbb{P}_1}(-1) \in \mathrm{fib}(\mathrm{K}(\mathbb{P}^1) \xrightarrow{i_\infty^*} \mathrm{K}(\{\infty\}))$$

Bott periodicity in particular tells us that the map

$$\beta\cdot\colon \mathrm{KGL}^{2,1}(\mathrm{Pic}) \to \mathrm{KGL}^{0,0}(\mathrm{Pic})$$

is an equivalence, whence we can lift (i.e. multiply with β^{-1}) the class $\mathscr{O} - \mathscr{L}^{\mathrm{univ}}$ in the target to a class $c_1(\mathscr{L}^{\mathrm{univ}})$ in the source, and it is easy to see that this defines an orientation. Indeed, restriction to $\mathbb{P}^1$ gives

$$c_1(\mathscr{L}^{\mathrm{univ}})\mid_{\mathbb{P}^1} = \beta^{-1}(\mathscr{O}_{\mathbb{P}_1} - \mathscr{L}^{\mathrm{univ}}\mid_{\mathbb{P}-1}) = \beta^{-1}(\mathscr{O}_{\mathbb{P}^1} - \mathscr{O}_{\mathbb{P}^1}(-1)) = \beta^{-1}\beta = 1.$$

Example 4.11. In fact, one can also compute the formal group law over $\mathrm{KGL}^*(S)$ explicitly. We leave it as an exercise to the reader to show that it is given by $f(x, y) = x + y \pm \beta x y$.

5 Universal property of K-theory

This talk follows Annala–Iwasa 2023. Let us fix some notation.

- $\mathscr{K}(X)$ is the K-theory anima of X.
- $\mathrm{K}^B(X)$ is the nonconnective K-theory spectrum of X, so that $\Omega^\infty \mathrm{K}^B(X) \simeq \mathscr{K}(X)$.
- KGL is the motivic spectrum representing nonconnective K-theory, so that $\Omega^\infty_{\mathbb{P}^1}\mathrm{KGL} = \mathrm{K}^B$ as sheaves of spectra.

Remark 5.1. The universal property of K-theory will be something like a motivic Snaith theorem. For this to work, note that we have an inclusion of animæ

$$e\colon \mathrm{Pic} \to \mathscr{K}$$

given by the inclusion of line bundles. Beware that this does not respect the additive $\mathbb{E}_\infty$-structure on $\mathscr{K}$, only the multiplicative one given by the tensor product of perfect complexes.

Proposition 5.2. *Let $E \in \mathrm{CAlg}(\mathrm{hMS}_S)$ be orientable, and M an E-module. Then the diagonal arrow in the diagram*

$$\begin{tikzcd} {[\mathscr{K}, \Omega^\infty \Omega^\infty_{\mathbb{P}^1} M]} \arrow[r, "e^*"] & {[\mathrm{Pic}, \Omega^\infty \Omega^\infty_{\mathbb{P}^1} M]} \\ {[\mathscr{K}, \Omega^\infty \Omega^\infty_{\mathbb{P}^1} M]^+} \arrow[u] \arrow[ur] & \end{tikzcd}$$

is an equivalence. The vertical arrow is the inclusion of homotopy classes of additive maps for the additive $\mathbb{E}_\infty$-structure on these sheaves of animæ.

Corollary 5.3. *Setting $M = \mathrm{Pic}_+ \otimes E$ in the Proposition above, we see that the map*

$$\mathrm{Pic}_+ \otimes E \xrightarrow{e \otimes E} \mathscr{K}_+ \otimes E$$

admits a retraction.

Example 5.4. When $M = \mathrm{KGL}$ this gives us Adams operations up to homotopy. Indeed, for all $n \in \mathbb{Z}$ we have maps

$$\psi_n\colon \mathrm{Pic} \to \mathscr{K}, \mathscr{L} \mapsto \mathscr{L}^{\otimes n}.$$

By this proposition they extend uniquely to additive maps

$$\psi_n\colon \mathscr{K} \to \mathscr{K}.$$

Remark 5.5. Nowadays we know that Adams operations can be extended to $\mathbb{E}_\infty$ ring maps once we invert n. The homotopy classes of maps we constructed above do not contain any $\mathbb{E}_\infty$ maps unless we invert n. We will see later how to construct these refined versions.

Proof. Let us denote by $\widetilde{\mathscr{K}} = \mathrm{fib}(\mathscr{K} \to \underline{\mathbb{Z}})$ the fibre of the rank map. By Quillen's plus construction machinery, we can identify $\widetilde{\mathscr{K}} \simeq L_{\mathrm{Zar}}\mathrm{BGL}^+$. In fact, the free module functor gives a splitting up to homotopy $\underline{\mathbb{Z}} \to \mathscr{K}$ of the rank map, so that we can write

$$\mathscr{K} \cong \widetilde{\mathscr{K}} \oplus \underline{\mathbb{Z}} \in \mathrm{CMon}^{\mathrm{gp}}(\mathrm{h}\mathscr{P}(\mathrm{Sm}_S)).$$

This gives rise to maps of split short exact sequences

$$\begin{tikzcd} {[\widetilde{\mathscr{K}}, \Omega^\infty \Omega^\infty_{\mathbb{P}^1} M]^+} \arrow[r] \arrow[d] & {[\mathscr{K}, \Omega^\infty \Omega^\infty_{\mathbb{P}^1} M]^+} \arrow[r] \arrow[d] & {[\underline{\mathbb{Z}}, \Omega^\infty \Omega^\infty_{\mathbb{P}^1} M]^+} \arrow[d] \\ {[\mathrm{Pic}, \Omega^\infty \Omega^\infty_{\mathbb{P}^1} M]_*} \arrow[r] & {[\mathrm{Pic}_+, \Omega^\infty \Omega^\infty_{\mathbb{P}^1} M]_*} \arrow[r] & {[S, \Omega^\infty \Omega^\infty_{\mathbb{P}^1} M]_*} \end{tikzcd}$$

It is clear that the right vertical map is an isomorphism, so it suffices to show that the left vertical map is an isomorphism. Using Remark 5.6 below, since $\Omega^\infty\Omega^\infty_{\mathbb{P}^1}M$ is an $\mathbb{E}_\infty$-group, we see that the source can be written as a subset of

$$[\widetilde{\mathscr{K}}, \Omega^\infty\Omega^\infty_{\mathbb{P}^1}M] \cong [\mathrm{BGL}, \Omega^\infty\Omega^\infty_{\mathbb{P}^1}M] \cong M^0(S)[\![c_1, c_2, \ldots]\!].$$

By the projective bundle formula we can write the target as

$$[\mathrm{Pic}, \Omega^\infty\Omega^\infty_{\mathbb{P}^1}M]_* \cong M^0(S)[\![c]\!]/M^0(S),$$

with the quotient on the right hand side ensuring we get pointed maps. Since e is the inclusion of line bundles, it is clear that the vertical map sends c_1 to c and every other c_i to zero. Let us now describe the subset of additive maps in the source and see that this maps bijectively to the target.

Let $f\colon \widetilde{\mathscr{K}} \to M$ be a map. Then additivity of f is the requirement that the diagram

$$\begin{tikzcd}
\widetilde{\mathscr{K}} \times \widetilde{\mathscr{K}} \arrow[r, "+"] \arrow[d, "f\times f"] & \widetilde{\mathscr{K}} \arrow[d, "f"] \\
M \times M \arrow[r, "+"] & M
\end{tikzcd}$$

commutes. On cohomology, we obtain a comultiplication

$$\Delta_M\colon M^0(S)[\![c_i \mid i \geq 1]\!] \to M^0(S)[\![c_i, c_i' \mid i \geq 1]\!].$$

Then multiplicativity of f viewed as a class in M-cohomology is precisely the requirement that

$$\Delta_M(f^*) = f^* \otimes 1 + 1 \otimes f^*.$$

Note that we have implicitly assumed that this comultiplication is part of a Hopf algebra structure, which can be done without loss of generality as Δ_M is the base change of Δ_E. Now recall the Whitney sum formula which states that

$$\Delta_E(c_n) = \sum_{p+q=n} c_p \otimes c_q'.$$

We conclude that

$$[\widetilde{\mathscr{K}}, \Omega^\infty\Omega^\infty_{\mathbb{P}^1}E] \cong \mathrm{Prim}^{\Delta_E} E^*(S)[\![c_i \mid i \geq 1]\!]^{\Sigma_\infty} \cong E^*(S)[\![c]\!].$$

□

Remark 5.6. Recall that the plus construction is such that the map $\mathrm{BGL} \to \mathrm{BGL}^+$ is acyclic, i.e. extends to a pushout

$$\begin{tikzcd}
\mathrm{BGL} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathrm{BGL}^+ \arrow[d, "\mathrm{id}"] \\
\mathrm{BGL}^+ \arrow[r] & \mathrm{BGL}^+.
\end{tikzcd}$$

This means that this map becomes an equivalence after applying any excisive functor, in particular, stably. For this reason, we will not need to distinguish between BGL and BGL^+ stably.

We can now proceed to prove a more structured version of this result, which will give us an identification of $\mathbb{E}_\infty$-rings as in Snaith' theorem.

Theorem 5.7 (Motivic Snaith theorem). *The map of $\mathbb{E}_\infty$-rings*

$$\Sigma^\infty_{\mathbb{P}^1}\Sigma^\infty\mathrm{Pic}_+ \to \mathrm{KGL}$$

factors through a map

$$\Sigma^\infty_{\mathbb{P}^1}\Sigma^\infty\mathrm{Pic}_+[\beta^{-1}] \to \mathrm{KGL}$$

and the latter is an equivalence.

Example 5.8. Using this more refined result we can then give $\mathbb{E}_\infty$ Adams operations. Indeed, we still have the $\mathbb{E}_\infty$-maps

$$\psi_n \colon \mathrm{Pic} \to \mathrm{Pic}, \mathscr{L} \mapsto \mathscr{L}^{\otimes n}$$

for any $n \in \mathbb{Z}$. Now we see that the induced map on $\mathbb{E}_\infty$-ring spectra

$$\psi_n \colon \Sigma^\infty_{\mathbb{P}^1} \Sigma_\infty \mathrm{Pic}_+ \to \Sigma^\infty_{\mathbb{P}^1} \Sigma^\infty \mathrm{Pic}_+$$

carries β to $n\beta$. By the theorem above this descends to a map of $\mathbb{E}_\infty$-rings

$$\psi_n \colon \mathrm{KGL}[1/n] \to \mathrm{KGL}[1/n].$$

Let us denote $\Omega^\infty \Sigma^\infty \mathrm{Pic}_+$ by $Q\mathrm{Pic} \in \mathscr{C} := \mathscr{P}(\mathrm{Sm}_S; \mathrm{Ani}_*)_{\mathrm{Nis,ebe}}$. Recall that we can construct

$$\beta = \mathscr{O} - \mathscr{O}(-1) \colon (\mathbb{P}^1, \infty) \to Q\mathrm{Pic},$$

and can therefore view this as a map to the unit in $\mathrm{Mod}(\mathscr{C}; Q\mathrm{Pic})$. It therefore makes sense to ask whether $Q\mathrm{Pic}$-modules in $\mathscr{C}$ are β-periodic. In fact, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Mod}(\mathscr{C}; Q\mathrm{Pic}) \arrow[r, two heads, "L_\beta"] \arrow[d, "\Sigma^\infty_{\mathbb{P}^1}"] & P_\beta \mathrm{Mod}(\mathscr{C}; Q\mathrm{Pic}) \arrow[d, "\sim"] \\
\mathrm{Mod}(\mathrm{MS}^{\mathrm{un}}_S; Q\mathrm{Pic}) \arrow[r, two heads, "L_\beta"] \arrow[d, "\Sigma^\infty"] & P_\beta \mathrm{Mod}(\mathrm{MS}^{\mathrm{un}}_S; Q\mathrm{Pic}) \arrow[d, shift right] \\
\mathrm{Mod}(\mathrm{MS}_S; Q\mathrm{Pic}) \arrow[r, two heads, "L_\beta"] & P_\beta \mathrm{Mod}(\mathrm{MS}_S; Q\mathrm{Pic}) \arrow[u, shift right, "i"']
\end{tikzcd}$$

where we have marked all localisation functors. The top right vertical arrow is an equivalence by the fact that $\mathbb{P}^1$ acts invertibly on β-periodic $Q\mathrm{Pic}$-modules. Note that KGL was the motivic spectrum obtained from $\mathscr{K}$ by Bott periodicity, so we see that going over and down sends $\mathscr{K}$ to KGL. Commutativity then tells us that the unit map

$$\Sigma^\infty_{\mathbb{P}^1} \Sigma^\infty \mathscr{K} \to \mathrm{KGL}$$

becomes an equivalence after applying L_β.

Remark 5.9. We can now try to give a fake proof of the motivic Snaith theorem. Indeed, letting g be the unit map obtained above we have a commutative triangle in MS_S of the form

$$\begin{tikzcd}
\Sigma^\infty_{\mathbb{P}^1} \Sigma^\infty \mathrm{Pic}_+ \arrow[r, "f"] \arrow[d, "e"] & \mathrm{KGL} \\
\Sigma^\infty_{\mathbb{P}^1} \Sigma^\infty \mathscr{K}, \arrow[ur, "g"] &
\end{tikzcd}$$

so by two-out-of-three it suffices to show $L_\beta e$ is an equivalence. But now just apply the Yoneda lemma and make recursion to Proposition 5.2 using that the sources are already oriented so it is sufficient to check the Yoneda argument on oriented targets. This proof is wrong for two reasons.

- The map e is not at all $Q\mathrm{Pic}$-linear, so it does not induce a map on L_β, as this functor is only well defined on $Q\mathrm{Pic}$-modules.
- In fact, one can show that
$$E^* L_\beta \Sigma^\infty_{\mathbb{P}^1} \Sigma^\infty \mathrm{Pic}_+ \cong E^* L_\beta \Sigma^\infty_{\mathbb{P}^1} \Sigma^\infty \mathscr{K}$$
for any oriented theory, but this isomorphism is just not induced by e.

It turns out that the $\mathbb{A}^1$-invariant literature overlooked this issue, and the proofs in Gepner–Snaith 2009 and Spitzweck–Østvær 2009 are simply wrong.

Notes for the rest of this lecture are incomplete

6 Constructing the J-homomorphism

Recall that we defined the Thom spectrum of finite locally free sheaf $\mathscr{E} \in \mathrm{Vect}(X)$ for $X \in \mathscr{P}(\mathrm{Sm}_S)$ as

$$\mathrm{Th}_X(\mathscr{E}) = \mathbb{P}(\mathscr{E} \oplus \mathscr{O})/\mathbb{P}(\mathscr{E}).$$

We want to prove in this section that this association is multiplicative.

Theorem 6.1. *Let S be a derived scheme. Then the assignment $\mathscr{E} \mapsto \mathrm{Th}_S(\mathscr{E}) \in \mathrm{MS}_S$ extends to an $\mathbb{E}_\infty$ map*

$$(\mathscr{K}(S), +) \to (\mathrm{Pic}(\mathrm{MS}_S), \otimes).$$

Remark 6.2. Note that in the $\mathbb{A}^1$-invariant setting, this theorem is trivial. Indeed, we then have a purity equivalence

$$\mathrm{Th}_X(\mathscr{E}) \simeq \mathbb{V}(\mathscr{E})/(\mathbb{V}(\mathscr{E}) \setminus X)$$

and we can use that for any open embeddings $U \hookrightarrow X$, $V \hookrightarrow Y$, there is a Zariski local equivalence

$$\frac{X}{U} \wedge \frac{Y}{V} \simeq \frac{X \times Y}{X \times V \cup_{U \times V} U \times Y} \simeq (X \times V) \cup (U \times Y).$$

To approach this theorem in the non-$\mathbb{A}^1$-invariant setting, we need to find a relation between a quotient of $\mathbb{P}(\mathscr{E} \oplus \mathscr{O}) \times \mathbb{P}(\mathscr{F} \oplus \mathscr{O})$ and $\mathbb{P}(\mathscr{E} \oplus \mathscr{F} \oplus \mathscr{O})$ for $\mathscr{E}$ and $\mathscr{F}$ finite locally free sheaves on S.

6.1 Derived blowups

The following proposition serves as a definition for either of the equivalent conditions below.

Proposition 6.3. *Let $f : Y \to X$ be a morphism of derived schemes. Then the following are equivalent.*

1. *Locally on Y, f can be factored as*
$$f : Y \to V \xrightarrow{p} X,$$
where p is smooth and Y is the vanishing locus $Y = Z(f_1, \ldots, f_m)$ of a collection of functions $f_i \in \mathscr{O}(V)$.
2. *Locally on Y, f is the base change of some map $\mathbb{A}^n_{\mathbb{Z}} \to \mathbb{A}^m_{\mathbb{Z}}$.*
3. *f is locally of finite presentation and the cotangent complex L_f is of Tor amplitude $[0, 1]$. Moreover, L_f is of rank $n - m$.*

Definition 6.4. We say f is quasi-smooth if either of the equivalent conditions of Proposition 6.3 are satisfied. A Cartier divisor is a quasi-smooth closed immersion of codimension one.

Definition 6.5. An excess intersection square is a commutative diagram

$$\begin{tikzcd} Z' \arrow[r, "e"] \arrow[d, "g"] & X' \arrow[d, "f"] \\ Z \arrow[r, "e"] & X \end{tikzcd}$$

where

- e and e' are closed immersions,
- the resulting map $Z' \to X' \times_X Z$ is surjective, and
- $g^* N_{Z/X} \to N_{Z'/X'}$ is surjective on π_0.

Recall that we identify $N_f \simeq L_f[-1]$.

Definition 6.6. Let $Z \to X$ be a closed immersion. Then we define the blowup of X at Z in terms of the functor of points as

$$\mathrm{Bl}_Z(X)(Y) = \{\text{excess intersection squares of the form } \begin{tikzcd} D \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & X \end{tikzcd} \text{ with } D \to Y \text{ a Cartier divisor}\}.$$

Proposition 6.7 (Hekking, Khan, Rydh). *$\mathrm{Bl}_Z(X)$ as defined above defines a derived scheme. In particular, given a closed embedding $Z \to X$ as above, there exists a universal excess intersection square*

$$\begin{tikzcd} E \arrow[r] \arrow[d] & \mathrm{Bl}_Z(X) \arrow[d] \\ Z \arrow[r] & X. \end{tikzcd}$$

A good supply of blowups comes from locally free sheaves. Let $\mathscr{E} \in \mathrm{Vect}(X)$ be a locally free sheaf with a map $\sigma\colon \mathscr{E} \to \mathscr{O}_X$, i.e. a section of $\mathbb{V}(\mathscr{E}) \to X$. We then define $Z(\sigma) \to X$ by means of the pullback diagram

$$\begin{tikzcd} Z(\sigma) \arrow[r] \arrow[d] & X \arrow[d, "\sigma"] \\ X \arrow[r, "0"] & \mathbb{V}(\mathscr{E}). \end{tikzcd}$$

We can then verify from the condition that the derived scheme $\mathrm{Bl}_{Z(\sigma)}X$ classified factorisations of the form

$$\begin{tikzcd} \mathscr{E} \arrow[rr, "\sigma"] \arrow[dr, "\phi"'] & & \mathscr{O} \\ & \mathscr{L} \arrow[ur, "\tau"'] & \end{tikzcd}$$

with $\mathscr{L} \in \mathrm{Pic}(X)$. The Cartier divisor in the excess intersection square classified by this factorisation is precisely $Z(\tau) \to X$.

Let us now return to our main goal of, part of which is to construct a symmetric monoidal functor, natural in the base, of the form

$$\mathrm{Th}_S\colon \mathrm{Vect}(S)^{\mathrm{epi}} \to \mathscr{P}(\mathrm{Sm}_S; \mathrm{Ani}_*)_{\mathrm{ebe}}$$

such that in particular $\mathrm{Th}_S(\mathscr{O}_S) = (\mathbb{P}^1, \infty)$. This will give rise to the multiplicative J-homomorphism by extension

$$\begin{tikzcd} \mathrm{Vect}(S)^{\simeq} \arrow[r] \arrow[d] & \mathrm{MS}_S \\ \mathscr{K}(S) \arrow[r, dashed, "\mathbb{E}_\infty"] & \mathrm{Pic}(\mathrm{MS}_S), \arrow[u, hook] \end{tikzcd}$$

where the top horizontal arrow sends $\mathscr{O}^n$ to $(\mathbb{P}^1, \infty)^{\otimes n}$. The first part of realising this consists of exhibiting an equivalence

$$\wedge_{i \in I} \mathrm{Th}_S(\mathscr{E}_i) \simeq \mathrm{Th}(\bigoplus_{i \in I} \mathscr{E}_i)$$

for any finite collection $\{\mathscr{E}_i\}_{i \in I}$ of locally free sheaves on S.

Definition 6.8. A smooth normal crossing divisor (SNCD) ∂X on a derived scheme X is a finite family of Cartier divisors

$$\partial X = (D_i \to X)_{i \in I}.$$

For any subset $J \subset I$, write $\partial_J X := \bigcap_{i \in J} D_i$. Further, if $X \in \mathrm{Sm}_S$, a relative SNCD ∂X on X is a SNCD such that each $\partial_J X$ is smooth over S for all subsets $J \subset I$.

These assemble to define a category $\mathrm{Sm}_S^{\mathrm{sncd}}$ whose objects are pairs $(X, \partial X)$ of a smooth S-scheme and a relative SNCD on X. We can define a functor

$$q\colon \mathrm{Sm}_S^{\mathrm{sncd}} \to \mathscr{P}(\mathrm{Sm}_S), (X, \partial X) \mapsto \mathrm{tcof}(J \mapsto \partial_J X)$$

in terms of the total cofibre. Further note that this category has products given by

$$(X, \partial X) \times (Y, \partial Y) = (X \times Y, \partial X \times Y \cup X \times \partial Y).$$

Example 6.9. The key example is the relative SNCD given by the section at infinity $\mathbb{P}(\mathscr{E}) \to \mathbb{P}(\mathscr{E} \oplus \mathscr{O})$. Essentially per construction, we have

$$q(\mathbb{P}(\mathscr{E} \oplus \mathscr{O}), \mathbb{P}(\mathscr{E})) = \mathrm{Th}_S(\mathscr{E}).$$

Given a finite collection $\mathscr{E} := \{\mathscr{E}_i\}_{i\in I}$ as earlier, we will construct $\mathbb{B}(\mathscr{E}) \in \mathrm{Sm}_S^{\mathrm{sncd}}$ and a zig-zag of maps

$$\prod_{i\in I} \mathbb{P}(\mathscr{E}_i \oplus \mathscr{O}) \xleftarrow{b_\Pi} q\mathbb{B}(\mathscr{E}) \xrightarrow{b_{\mathbb{P}}} \mathbb{P}(\bigoplus_{i\in I} \mathscr{E}_i \oplus \mathscr{O})$$

such that both b_Π and $b_{\mathbb{P}}$ become equivalences in $\mathscr{P}(\mathrm{Sm}_S; \mathrm{Ani}_*)_{\mathrm{sbe}}$.

Remark 6.10. The key point is the following; if $(X, \partial X)$ is a relative SNCD and $Z \to X$ is a closed embedding in Sm_S with Z contained in ∂X, then the map

$$(\mathrm{Bl}_Z(X), \widetilde{\partial X} \cup E) \to (X, \partial X)$$

becomes an equivalence in $\mathscr{P}_{\mathrm{sbe}}$

Example 6.11. When $I = \{1, 2\}$, and we assume that $\mathscr{E}_1 =_2= \mathscr{O}$, then b_Π has target $\mathbb{P}^1 \times \mathbb{P}^1$, i.e. the blowup in one point of $\mathbb{P}(\mathscr{E}_1) \times \mathbb{P}(\mathscr{E}_1)$, while $b_{\mathbb{P}}$ has target $\mathbb{P}^2$, i.e. the blowup in two points of $\mathbb{P}(\mathscr{E}_1) \sqcup \mathbb{P}(\mathscr{E}_2)$.

In general, $\mathbb{B}(\mathscr{E})$ classifies epimorphisms of I-cubes of the form

$$(\phi_J)_j\colon (\bigoplus_{i\in J} \mathscr{E}_i \oplus \mathscr{O})_{J\subset I} \to (\mathscr{L}_J)_{J\subset I},$$

where each $\mathscr{L}_J$ is invertible. b_Π then sends such a collection to the collection of epimorphisms $(\phi_{\{i\}})_{i\in I}$, while $b_{\mathbb{P}}$ sends it to the single epimorphism ϕ_I.

Theorem 6.12. *the morphisms b_Π and $b_{\mathbb{P}}$ defined above can both be written as sequences of smooth blowups.*

Definition 6.13. Let $\mathbb{B}(\mathscr{E})^\vee$ classify epimorphisms $\mathscr{E}_i \oplus \mathscr{O} \to \mathscr{L}_i$ for every $i \in I$ together with monomorphisms of punctures cubes under $\mathscr{O}$

$$0 \to (\mathscr{M}_K)_{K\subset I, K\neq\varnothing} \hookrightarrow (\bigoplus_{i\in K} \mathscr{L}_i)_{K\subset I, K\neq\varnothing}$$

where every $\mathscr{M}_K$ is invertible.

Proposition 6.14. *Let $\mathscr{C}$ be a stable category, and let $P = \mathscr{P}(I) \setminus \{I\}$ be the poset of strict subsets of I for I some finite set. Then there is an equivalence*

$$\mathrm{Fun}(P, \mathscr{C}) \xrightarrow{\sim} \mathrm{Fun}(P^{\mathrm{op}}, \mathscr{C})$$

that sends F to the functor $p \mapsto \varinjlim_{p\to q} F(q)$ with functoriality inherited from the indexing category being the slice $P_{p/}$.

Example 6.15. If I has two elements, P is the span category, so this gives an equivalence between spans and cospans in $\mathscr{C}$, which is immediate from the stability in $\mathscr{C}$.

7 Algebraic cobordism and the universal orientation

Let us recall the definition of algebraic cobordism.

Definition 7.1. Define the algebraic cobordism spectrum $\mathrm{MGL} \in \mathrm{CAlg}(\mathrm{MS}_S)$ by

$$\mathrm{MGL} := \varinjlim_{(X,\xi)} \mathrm{Th}_X(\xi) \in \mathrm{CAlg}(\mathrm{MS}_S),$$

where the indexing category consists of pairs of a smooth S-scheme X and a K-theory class ξ on X of rank zero.

More precisely, we use the $\mathbb{E}_\infty$-map $\mathscr{K} \to \mathrm{MS}$ given by the J-homomorphism to get a symmetric monoidal functor from the unstraightening

$$\begin{tikzcd} \int_{\mathrm{Sm}_S} \mathscr{K}^0 \arrow[r] \arrow[d] & \mathrm{Ani}_*^{\mathrm{op}} \arrow[d] \\ \mathrm{Sm}_S \arrow[r] & \mathrm{Ani}^{\mathrm{op}} \end{tikzcd}$$

of the form

$$J\colon \int_{\mathrm{Sm}_S} \mathscr{K}^0 \to \mathrm{MS}_S, (X,\xi) \mapsto \mathrm{Th}_X(\xi).$$

The fact that this functor is symmetric monoidal then gives us the $\mathbb{E}_\infty$-ring structure on its colimit for free. Note that one could have dropped the restriction to rank zero K-theory classes as well.

Definition 7.2. Define periodic algebraic cobordism PMGL by

$$\mathrm{PMGL} := \varinjlim\Big(\int_{\mathrm{Sm}_S} \mathscr{K} \to \mathrm{MS}_S\Big).$$

However, note that the symmetric monoidal inclusion $\mathscr{K}^0 \to \mathscr{K}$ incudes an $\mathbb{E}_\infty$-map $\mathrm{MGL} \to \mathrm{PMGL}$ and in fact this is part of an equivalence

$$\mathrm{PMGL} \simeq \bigoplus_{n\in\mathbb{Z}} \Sigma^n_{\mathbb{P}^1}\mathrm{MGL}.$$

Remark 7.3. The right hand side of this equivalence does not naturally have an $\mathbb{E}_\infty$-structure, e.g. the periodic motivic sphere spectrum is not $\mathbb{E}_\infty$, so this is something special about MGL, namely that fact that the K-theory anima is the direct sum of a graded commutative $\mathbb{Z}$-graded anima $\{\mathscr{K}^n\}_{n\in\mathbb{Z}}$ of rank n K-theory classes.

Recall that we defined $\mathrm{Gr}_\infty := \varinjlim_n \mathrm{Gr}_n$ and $\mathrm{BGL}_\infty := \varinjlim_n \mathrm{BGL}_n$, where BGL_n is also known as Vect_n, the moduli of rank n vector bundles. There are natural maps

$$\mathrm{Gr}_\infty \to \mathrm{BGL}_\infty \to \mathscr{K}^0$$

where the latter map sends $\mathscr{E} \in \mathrm{BGL}_n$ to the class $[\mathscr{E}] - [\mathscr{O}^n]$. We could then take the colimit after precomposing with these maps to obtain new Thom spectra. We will see that these are all equivalent.

Theorem 7.4 (A). *Both of the maps $\mathrm{Gr}_\infty \to \mathscr{K}^0$ and $\mathrm{BGL}_\infty \to \mathscr{K}^0$ induce equivalences on Thom spectra in MS_S. In particular, this gives us equivalences*

$$\mathrm{MGL} \simeq \varinjlim_n \Sigma^{-n}_{\mathbb{P}^1}\mathrm{Th}_{\mathrm{BGL}_n}(\mathscr{E}^n), \tag{A1}$$

$$\mathrm{MGL} \simeq \varinjlim_n \Sigma^{-n}_{\mathbb{P}^1}\mathrm{Th}_{\mathrm{Gr}_n}(Q_n), \tag{A2}$$

where $\mathscr{E}_n$ and Q_n are the universal rank n bundle and universal quotient respectively.

We can use this description to obtain the universal property of MGL with respect to orientations.

Theorem 7.5. MGL *is the initial oriented object in* $\mathrm{CAlg}(\mathrm{hMS}_S)$*. This means that for any homotopy commutative ring spectrum* E *there is a bijection between the set of homotopy ring maps* $\mathrm{MGL} \to E$ *and the set of orientations of* E*.*

Let us begin by proving (A1).

Proof of (A1). If X is an affine derived scheme, Quillen's plus construction tells us that the map

$$\mathrm{Vect}_\infty(X) = \mathrm{BGL}_\infty(X) \to \mathscr{K}(X)^0$$

is acyclic, i.e. an epimorphism of anima. By descent, this means that the map of $\mathrm{BGL}_\infty \to \mathscr{K}^0$ is an epimorphism in the Zariski topos, i.e. there is a pushout square of sheaves

$$\begin{tikzcd} \mathrm{BGL}_\infty \arrow[r] \arrow[d] & \mathscr{K}^0 \arrow[d, "\mathrm{id}"] \\ \mathscr{K}^0 \arrow[r, "\mathrm{id}"] & \mathscr{K}^0. \end{tikzcd}$$

By the J-homomorphism $\mathscr{K}^0 \to \mathrm{MS}_S^{\simeq}$ viewed as a map of Zariski sheaves, this square lives in $\mathscr{P}(\mathrm{Sm}_S)_{/\mathrm{MS}^{\simeq}}$, i.e. the category of presheaves $F \in \mathscr{P}(\mathrm{Sm}_S)$ with a map $\int_{\mathrm{Sm}_S} F \to \int_{\mathrm{Sm}_S} \mathrm{MS}^{\simeq}$. By the formal construction of the J-homomorphism, as well as descent, we obtain a factorisation

$$\begin{tikzcd} \mathscr{P}(\mathrm{Sm}_S)_{/\mathrm{MS}^{\simeq}} \arrow[r] \arrow[dr] & \mathrm{MS}_S \\ & \mathcal{S}\mathrm{hv}_{\mathrm{Nis}}(\mathrm{Sm}_S), \arrow[u] \end{tikzcd}$$

where the top horizontal arrow is the relative colimit (in fact the colimit internal to the topos $\mathscr{P}(\mathrm{Sm}_S)$) and the resulting vertical arrow preserves colimits. Applying the resulting functor to the pushout square above gives a pushout square in MS_S of the form

$$\begin{tikzcd} \varinjlim_n \Sigma^{-n}_{\mathbb{P}^1} \mathrm{Th}_{\mathrm{BGL}_n}(\mathscr{E}_n) \arrow[r] \arrow[d] & \mathrm{MGL} \arrow[d, "\mathrm{id}"] \\ \mathrm{MGL} \arrow[r, "\mathrm{id}"] & \mathrm{MGL}. \end{tikzcd}$$

However, we are now in a stable category so that this is equivalently a pullback square and we conclude that the top horizontal map is an equivalence. □

To prove Theorem B (7.5), we reduce this statement to two points.

B1 MGL is oriented.

B2 If $E \in \mathrm{CAlg}(\mathrm{hMS}_S)$ is oriented, then there exists a unique ring map $\mathrm{MGL} \to E$ preserving the orientation.

Our approach to orientations in proving either statement will use the fact that we can obtain orientations from Thom isomorphisms.

Remark 7.6. It is clear from the definition that MGL has Thom isomorphisms. Indeed, letting Y be a smooth scheme over S and $\eta \in \mathscr{K}^0(Y)$ a rank zero K-theory class we obtain

$$\mathrm{MGL} \otimes \mathrm{Th}_Y(\eta) \simeq \varinjlim_{(X,\xi)} \mathrm{Th}_X(\xi) \otimes \mathrm{Th}_Y(\eta) \simeq \varinjlim_{(X',\xi')} \mathrm{Th}_{X'}(\xi') \simeq \mathrm{MGL} \otimes \mathbb{1},$$

where we just used the multiplicativity of Thom spectra to reindex our colimit to a cofinal subsystem.

Proof of B2. Note that there is an inclusion

$$\mathrm{Hom}_{\mathrm{CAlg}(\mathrm{hMS}_S)}(\mathrm{MGL}, E) \subset E^0(\mathrm{MGL}).$$

Since E was assumed to be oriented, we can use the Thom isomorphism to identify the right hand side as

$$\begin{aligned} E^0(\mathrm{MGL}) &\cong E^0(\mathrm{BGL}_\infty), \\ &\cong \varprojlim_n E^0(\mathrm{BGL}_n), \\ &\cong (E^*(S)[\![c_1, c_2, \ldots]\!])^0. \end{aligned}$$

In this identification, we further used the Milnor sequence to bring out the colimit defining BGL_∞, noting that all transition maps in this system are surjective by our computation of the oriented cohomology of BGL_n. The element $1 \in \cong (E^*(S)[\![c_1, c_2, \ldots]\!])^0$ correponds to an element we call t in $E^0(\mathrm{MGL})$.

We claim that $t \in E^0(\mathrm{MGL})$ in fact lands in the subset $\mathrm{Hom}_{\mathrm{CAlg}(\mathrm{hMS}_S)}(\mathrm{MGL}, E)$. First, note that t is uniquely determined by commutativity of the diagram

$$\begin{tikzcd} \mathrm{MGL} \arrow[r, "t"] & E \\ \mathrm{Th}_{\mathrm{BGL}_n}(\mathscr{E}_n - \mathscr{O}^n) \arrow[u, "t_{\mathrm{MGL}}"] \arrow[ur, "t_n"'] & \end{tikzcd}$$

for all n, where t_{MGL} is the Thom class of MGL and t_n is the Thom class of $\mathscr{E}_n$ in E-cohomology. Therefore, showing that t is a ring map boils down to commutativity of the diagrams

$$\begin{tikzcd} \mathrm{Th}_{\mathrm{BGL}_n}(\mathscr{E}_n - \mathscr{O}^n) \otimes \mathrm{Th}_{\mathrm{BGL}_m}(\mathscr{E}_m - \mathscr{O}^m) \arrow[r, "t_n \otimes t_m"] \arrow[d] & E \otimes E \arrow[d, "\mu_E"] \\ \mathrm{Th}_{\mathrm{BGL}_{m+n}}(\mathscr{E}_{n+m} - \mathscr{O}^{n+m}) \arrow[r, "t_{n+m}"] & E. \end{tikzcd}$$

Noting that the multiplicative structure on MGL can be described in terms of the direct sum maps $\mathrm{BGL}_n \times \mathrm{BGL}_m \to \mathrm{BGL}_{n+m}$, this will follows from the lemma below. □

Lemma 7.7. *For E an oriented homotopy commutative ring spectrum and $X, Y \in \mathscr{P}(\mathrm{Sm}_S)$ with finite locally free sheaves $\mathscr{E} \in \mathrm{Vect}(X)$, $\mathscr{F} \in \mathrm{Vect}(Y)$, there is an isomorphism*

$$E^*(\mathrm{Th}_{X \times Y}(\mathscr{E} \boxtimes \mathscr{F})) \cong E^*(\mathrm{Th}_X(\mathscr{E}) \wedge \mathrm{Th}_Y(\mathscr{F}))$$

that takes the Thom class $t(\mathscr{E} \boxtimes \mathscr{F})$ on the left hand side to the product $t(\mathscr{E})t(\mathscr{F})$ on the right hand side.

Proof. This is an immediate consequence of the Whitney sum formula for Chern classes, and is in fact equivalent to it. □

7.1 Orientations via Thom classes

This section is devoted to construction orientations for motivic spectra that have Thom classes. This will essentially prove part B1 of Theorem 7.5.

Remark 7.8. Let $\mathrm{Pic} \in \mathscr{P}(\mathrm{Sm}_S)$ be the Picard anima, and let $\mathscr{L} \in \mathrm{Pic}(\mathrm{Pic})$ be the universal line bundle. Then recall that the datum of an orientation was determined by the datum of a first Chern class $c_1(\mathscr{L})$, and this could be used to define Thom isomorphisms.

Since $\mathrm{Th}_{\mathrm{Pic}}(\mathscr{L})$ is defined as the cofibre of $\mathbb{P}_{\mathrm{Pic}}(\mathscr{L} \oplus \mathscr{O}) \to \mathbb{P}_{\mathrm{Pic}}(\mathscr{L})$, we can describe the Thom isomorphism for $\mathscr{L}$ in terms of the Chern classes as a map to the fibre

$$\begin{tikzcd}
 & E^{\mathrm{Th}_{\mathrm{Pic}}(\mathscr{L})} \arrow[d] \\
\Sigma_{\mathbb{P}^1} E^{\mathrm{Pic}} \arrow[r, "c_1(\mathscr{O}(1)) - c_1(\mathscr{L})"] \arrow[ur, dotted, bend left, "\sim"] & E^{\mathbb{P}_{\mathrm{Pic}}(\mathscr{L} \oplus \mathscr{O})_+} \arrow[d] \\
 & E^{\mathbb{P}_{\mathrm{Pic}}(\mathscr{L})_+},
\end{tikzcd}$$

where the vertical sequence is a fibre sequence.

Now note that the inclusion of the point $* \to \mathrm{Pic}$ induces a map

$$\mathbb{P}^1 = \mathrm{Th}_*(\mathscr{L}\mid_*) \to \mathrm{Th}_{\mathrm{Pic}}(\mathscr{L}),$$

hence a map

$$E^{\mathrm{Th}_{\mathrm{Pic}}(\mathscr{L})} \to \mathbb{E}^{\mathbb{P}^1}$$

such that the composite

$$\Sigma_{\mathbb{P}^1} E \to \Sigma_{\mathbb{P}^1} E^{\mathrm{Pic}} \simeq E^{\mathrm{Th}_{\mathrm{Pic}}(\mathscr{L})} \to E^{\mathbb{P}^1}$$

is the identity. Therefore, just the datum of the first Chern class of $\mathscr{L}$ and corresponding Thom isomorphism gives us a retraction of the map

$$\mathbb{P}^1 \otimes E \to \mathrm{Th}_{\mathrm{Pic}}(\mathscr{L}) \otimes E.$$

Remark 7.9. This is to be compared to our definition of an orientation as a retraction of the map

$$\mathbb{P}^1 \otimes E \to \mathrm{Pic} \otimes E.$$

In fact, we will see that they are equivalent.

Proposition 7.10. *Let E be a motivic spectrum, then there is a bijection of the form*

$$\{\textit{retractions of } \mathbb{P}^1 \otimes E \to \mathrm{Pic} \otimes E\} \cong \{\textit{retractions of } \mathbb{P}^1 \otimes E \to \mathrm{Th}_{\mathrm{Pic}}(\mathscr{L}) \otimes E\},$$

where the left hand side is the set of possible orientations of E.

Remark 7.11. Note that the map from left to right was essentially constructed above by means of the Thom isomorphism.

The map from right to left is given by

$$t \mapsto -t \circ \widetilde{s_0},$$

where $\widetilde{s_0}\colon \mathrm{Pic} \to \mathrm{Th}_{\mathrm{Pic}}(\mathscr{L})$ is the reduced zero section, i.e. the map

$$\mathrm{Pic} \xrightarrow{0} \mathbb{P}_{\mathrm{Pic}}(\mathscr{L} \oplus \mathscr{O}) \to \mathbb{P}_{\mathrm{Pic}}(\mathscr{L} \oplus \mathscr{O})/\mathbb{P}_{\mathrm{Pic}}(\mathscr{L} \oplus \mathscr{O}) = \mathrm{Th}_{\mathrm{Pic}}(\mathscr{L}).$$

Remark 7.12. The minus sign is there to be compatible with the definition of the top Chern class. Recall that this was given by pulling back the Thom class along any section up to a sign given by the rank of the bundle in question. Since $\mathscr{L}$ is a line bundle this sign is -1.

Proof of Proposition 7.10. The proof is an almost entirely straightforward consequence of the computation of $E^*\mathrm{Pic}$. The only nontrivial part consists of showing that $-t \circ \widetilde{s_0}$ is in fact an orientation. Indeed, consider the diagram

$$\begin{tikzcd}
\mathrm{Pic} \otimes E \arrow[r, "-\widetilde{s_0}"] & \mathrm{Th}_{\mathrm{Pic}}(\mathscr{L}) \otimes E \arrow[r, "t"] & \mathbb{P}^1 \otimes E \\
\mathbb{P}^1 \otimes E \arrow[u, "\mathscr{O}(1)"] \arrow[r, "-\widetilde{s_0}"] \arrow[dr, bend right, "\mathrm{id}"'] & \mathrm{Th}_{\mathbb{P}^1}(\mathscr{O}(1)) \otimes E \arrow[u] & \\
 & \mathrm{Th}_*(\mathscr{O}(1)\mid_*) \arrow[u, "i"] \arrow[r, "\sim"] & \mathbb{P}_1 \otimes E, \arrow[uu, bend right, "\mathrm{id}"']
\end{tikzcd}$$

where i is the map induced by the inclusion $* \xrightarrow{\infty} \mathbb{P}^1$ of the point at infinity. It is clear that both squares in this diagram commute, essentially per construction, and that $-t \circ \widetilde{s_0}$ is an orientation if and only if the boundary of diagram commutes, i.e. if and only if the bottom left triangle commutes. We have therefore reduced this problem to finding a homotopy

$$i \simeq -\widetilde{s_0}.$$

This is the content of Proposition 7.13 below. □

Proposition 7.13. *Using the same notation as in the proof above, the two maps*

$$\mathbb{P}^1_+ \underset{i}{\overset{-\widetilde{s_0}}{\rightrightarrows}} \mathrm{Th}_{\mathbb{P}^1}(\mathscr{O}(1))$$

are homotopic.

Remark 7.14. Compare the $\mathbb{A}^1$-invariant statement given by Panin and Smirnov.

Proof. Recall that per definition $\mathrm{Th}_{\mathbb{P}^1}(\mathscr{O}(1)) = \mathbb{P}_{\mathbb{P}^1}(\mathscr{O}(1) \oplus \mathscr{O})/\mathbb{P}_{\mathbb{P}^1}(\mathscr{O}(1))$. This quotient fits into a blowup square

$$\begin{tikzcd}
\mathbb{P}^1 \arrow[r, shift left, "s_0"] \arrow[d] & \mathbb{P}_{\mathbb{P}^1}(\mathscr{O}(1) \oplus \mathscr{O}) \arrow[l, shift left, "p"] \arrow[r, "q"] \arrow[d, "b"] & \mathrm{Th}_{\mathbb{P}^1}(\mathscr{O}(1)), \\
* \arrow[r] & \mathbb{P}^2 &
\end{tikzcd}$$

where p is the projection associated to this projective bundle, s_0 is its zero section, and q is the canonical map to the quotient. Restriction to the point at infinity in the $\mathbb{P}^1 = \mathbb{P}_{\mathbb{P}^1}(\mathscr{O}(1))$ on the top left, we see that this extends to a larger commutative diagram

$$\begin{tikzcd}
\infty \arrow[r, "0"] \arrow[d] & \mathbb{P}^1 \arrow[d, "i"] & \\
\mathbb{P}^1 \arrow[r, shift left, "s_0"] \arrow[d] & \mathbb{P}_{\mathbb{P}^1}(\mathscr{O}(1) \oplus \mathscr{O}) \arrow[l, shift left, "p"] \arrow[r, "q"] \arrow[d, "b"] & \mathrm{Th}_{\mathbb{P}^1}(\mathscr{O}(1)), \\
* \arrow[r] & \mathbb{P}^2 &
\end{tikzcd}$$

with the map i from the Proposition. Now define

$$y := \mathrm{id} - s_0 \circ p$$

as a self-map of $Y_+ := \mathbb{P}_{\mathbb{P}_1}(\mathscr{O}(1) \oplus \mathscr{O})_+$ in MS_S. Note that s_0 is a section of p so it is clear that

$$\begin{aligned}
y \circ s_0 &= s_0 - s_0 \circ p \circ s_0, \\
&= s_0 - s_0, \\
&= 0
\end{aligned}$$

as a map $\mathbb{P}^1_+ \to Y_+$. Therefore, y factors through the cofibre, i.e. there is a map $\overline{y}\colon \mathbb{P}^2_+ \to Y_+$ such that

$$y \simeq \overline{y} \circ b.$$

Now note that $b \circ i$ and $b \circ s_\infty$, for s_∞ the section at infinity of p, are both linear embeddings from $\mathbb{P}^1$ to $\mathbb{P}^2$. Projective homotopy invariance in MS_S therefore guarantees that these are homotopic. We conclude that

$$\begin{aligned} y \circ i &\simeq \overline{y} \circ b \circ i, \\ &\simeq \overline{y} \circ b \circ s_\infty, \\ &\simeq y \circ s_\infty. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} q \circ s_0 &\simeq q \circ (s_0 - s_\infty), \\ &\simeq -q \circ y \circ s_\infty, \\ &\simeq -q \circ y \circ i, \\ &\simeq -q \circ (i - s_0 \circ p \circ i), \\ &\simeq -q \circ i + q \circ s_0 \circ \infty \end{aligned}$$

In these identifications, we first used that $q \circ s_\infty$ since the quotient map is precisely collapsing the section at infinity, that $y \circ s_\infty = s_\infty - s_0 \circ p \circ s_\infty \simeq s_\infty - s_0$ by the section property, our observation that $y \circ s_\infty \simeq y \circ i$, the definition of y, and finally the fact that $p \circ i$ is constant at $\infty \in \mathbb{P}^1$. Now note that the points 0 and ∞ are homotopic in $\mathbb{P}^1$ are homotopic by projective homotopy invariance, so that $s \circ s_0 \circ \infty$ is null as q collapses the section at infinity. If we view the images of s_0 and s_∞ sitting inside $\mathbb{P}_{\mathbb{P}^1}(\mathscr{O}(1) \oplus \mathscr{O})$ as two projective lines, both of which intersect the projective line consisting of the image of i at the points ∞ and 0 respecitvely, we see that

$$s_0 \circ \infty = i \circ 0, \qquad\qquad i \circ \infty = s_\infty \circ \infty.$$

Therefore,

$$\begin{aligned} q \circ s_0 \circ \infty &= q \circ i \circ 0, \\ &\simeq q \circ i \circ \infty, \\ &\simeq q \circ s_\infty \infty, \\ &\simeq 0, \end{aligned}$$

once again since q collapses precisely the section at infinity. Bringing it all together we conclude that $-q \circ s_0 = -\widetilde{s_0}$ and i are homotopic. ◻

7.2 Grassmannian model for algebraic cobordism

This section is devoted to giving a sketch of the proof for part A2 of Theorem 7.4. Let

$$\mathrm{MGr} := \varinjlim_n \Sigma^{-n}_{\mathbb{P}^1} \mathrm{Th}_{\mathrm{Gr}_n}(Q_n)$$

be a motivic spectrum, and denote by

$$\phi\colon \mathrm{MGr} \to \varinjlim_n \Sigma^{-n}_{\mathbb{P}^1} \mathrm{Th}_{\mathrm{BGL}_n}(\mathscr{E}_n) \simeq \mathrm{MGL}$$

the map induced on colimits by the maps $\mathrm{Gr}_n \to \mathrm{BGL}_n$ classifying the universal quotient, implicitly postcomposed with the (multiplicative) equivalence of Theorem A1.

Remark 7.15. At this point, one might object that for every n, the map $\mathrm{Th}_{\mathrm{Gr}_n}(Q_n) \to \mathrm{Th}_{\mathrm{BGL}_n}(\mathscr{E}_n)$ induced by the map $\mathrm{Gr}_n \to \mathrm{BGL}_n$ should be an equivalence, since we showed that the latter map becomes an equivalence in MS_S. However, the Thom spectrum construction does not factor through motivic equivalences (in particular, it is constructed from unstable data, while motivic equivalences are equivalences of $\mathbb{P}^1$-spectra) so that this does not follow for free.

The strategy is as follows.

1. For any oriented $E \in \mathrm{CAlg}(\mathrm{hMS}_S)$, we show that ϕ induces an isomorphism in E^*. This is immediate, since E has Thom isomorphisms, so this reduces to showing that the map $E^*(\mathrm{BGL}_n) \to E^*(\mathrm{Gr}_n)$ is an equivalence, but E-cohomology factors through motivic equivalences so this follwos from our Grassmannian model for motivic classifying spaces of vector bundles.

2. MGr admits an orientation from its Thom classes. Indeed, our proof that Thom classes give rise to orientations took place entirely in MS_S, hence is invariant under the motivic equivalence $\mathbb{P}^\infty \simeq \mathrm{Pic}$. This argument then shows that it suffices for MGr to have a Thom isomorphism for $\mathrm{Th}_{\mathbb{P}^\infty}(\mathscr{O}(1))$ but under the identification $\mathbb{P}^\infty = \mathrm{Gr}_1$ this follows by the same trick of Thom isomorphisms for MGL by bringing it inside the colimit and reindexing over a cofinal subdiagram.

3. The map ϕ lifts to an $\mathbb{E}_\infty$ ring map. In particular, it is a map of homotopy commutative ring spectra, so that the universal property of MGL shows that ϕ must be an inverse to the homotopy commutative ring map $\mathrm{MGL} \to \mathrm{MGr}$ coming from the construction of orientations above, and we are done. This statement is the content of Proposition 7.16 below.

Proposition 7.16. *There exists an $\mathbb{E}_\infty$-ring structure on* MGr *such that the map $\phi\colon \mathrm{MGr} \to \mathrm{MGL}$ above is an $\mathbb{E}_\infty$-ring map.*

Proof. Given a finite locally free sheaf $\mathscr{E} \in \mathrm{Vect}(S)$, we obtain a symmetric sequence in the slice $(\mathrm{Sm}_S)_{/\mathrm{Vect}}$

$$\mathrm{Fin}^{\simeq} \to (\mathrm{Sm}_S)_{/\mathrm{Vect}}, I \mapsto \mathrm{Gr}_{|I|}(\mathscr{E}^I) \to \mathrm{Vect},$$

where the map to Vect above is the forgetful map. Furthermore, we can verify that this functor is lax symmetric monoidal, hence by the universal property of the Day convolution defines an $\mathbb{E}_\infty$ algebra in the functor category $\mathrm{SSeq}((\mathrm{Sm}_S)_{/\mathrm{Vect}})$. This enjoys good functoriality in $\mathscr{E}$ along epimorphisms (since this is the functorality that Grassmannians possess), hence assembles to a functor

$$\mathrm{Vect}^{\mathrm{epi}}(S)^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{SSeq}((\mathrm{Sm}_S)_{/\mathrm{Vect}})).$$

In particular, this sends the object $\mathscr{O}$ of the left hand side to the symmetric sequence $I \mapsto (\mathrm{Gr}_{|I|}(\mathscr{O}^I) \simeq S, \mathscr{O}^I)$, where the pair denotes the smooth S-scheme (in this case S itself) and the bundle on S classified by the map to Vect (in this case the product $\mathscr{O}^I$). Let us denote this $\mathbb{E}_\infty$-algebra in symmetric sequences by $(S, \mathscr{O}^-)$, then it is simple to verify that

$$(S, \mathscr{O}^-) \simeq \mathrm{Free}_{\mathbb{E}_\infty}(\varnothing, (S, \mathscr{O}), \varnothing, \varnothing, \ldots).$$

Let us now pass to slices under $\mathscr{O}$ (due to the op, this becomes a slice over $\mathscr{O}$ in the source) to obtain a functor

$$(\mathrm{Vect}^{\mathrm{epi}}(S)_{/\mathscr{O}})^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Mod}(\mathrm{SSeq}((\mathrm{Sm}_S)_{/\mathrm{Vect}}); (S, \mathscr{O}^-)).$$

Since the $\mathbb{E}_\infty$-algebra on the right hand side was free on the symmetric sequence $(\varnothing, (S, \mathscr{O}), \varnothing, \varnothing, \ldots)$, we can identify the right hand side further as

$$\begin{aligned}
(\mathrm{Vect}^{\mathrm{epi}}(S)_{/\mathscr{O}})^{\mathrm{op}} \to & \mathrm{CAlg}(\mathrm{Mod}(\mathrm{SSeq}((\mathrm{Sm}_S)_{/\mathrm{Vect}}); (S, \mathscr{O}^-)), \\
\simeq & \mathrm{CAlg}(\mathrm{Mod}(\mathrm{SSeq}((\mathrm{Sm}_S)_{/\mathrm{Vect}}); \mathrm{Free}_{\mathbb{E}_\infty}(\varnothing, (S, \mathscr{O}), \varnothing, \varnothing, \ldots)), \\
\simeq & \mathrm{CAlg}(\mathrm{Sp}^{\mathrm{lax}}_{(S,\mathscr{O})}((\mathrm{Sm}_S)_{/\mathrm{Vect}})), \\
\to & \mathrm{CAlg}(\mathrm{Sp}^{\mathrm{lax}}_{\mathbb{P}^1}\mathscr{P}(\mathrm{Sm}_S; \mathrm{Ani}_*)_{\mathrm{ebe}}), \\
\to & \mathrm{CAlg}(\mathrm{MS}_S).
\end{aligned}$$

The functors above are given by taking Thom spectra, followed by Nisnevich sheafification and spectrification. We see that under this composite, the bundle $\mathscr{O}^\infty$ gets sent to MGr, and that the slice over Vect tells us that the last functor in fact lifts to a slice $\mathrm{CAlg}(\mathrm{MS}_S)_{/\mathrm{MGL}}$. We have therefore constructed an $\mathbb{E}_\infty$-map $\mathrm{MGr} \to \mathrm{MGL}$, and chasing through the definitions shows that this agrees with ϕ on underling motivic spectra. ◻

8 The Conner–Floyd isomorphism

Theorem 8.1 (Conner–Floyd isomorphism). *Let S be a qcqs derived scheme, then there is an isomorphism*

$$\mathrm{K}^{*,*}(S) \cong \mathrm{MGL}^{*,*}(S) \otimes_L \mathbb{Z}[\beta^{\pm}],$$

where $K^{p,q}(S) = K_{2q-p}(S)$.

Remark 8.2. This is not particularly useful for computing algebraic K-theory, as the right hand side is very mysterious. However, upon rationalisation it does tell us that the Bousfield classes for algebraic K-theory, algebraic cobordism and motivic cohomology are the same.

Remark 8.3. This is a statement about abelian groups, and does not follow from some statement about motivic spectra. In fact, the right hand side does not admit a lift to a motivic spectrum (a priori). We will therefore have to rephrase the universal property of algebraic cobordism as an Ab-valued cohomology theory on motivic spectra. The universal property of the cohomology theory associated to KGL is simpler, since it follows from the universal property of K-theory which we proved earlier. We will not go through the details of setting up the universal property of MGL as a cohomology theory in these notes.

This proof goes through the theory of formal group laws. Recall that is $E \in \mathrm{CAlg}(\mathrm{hMS}_S)$ is an oriented, then we obtain a graded formal group law

$$f_E(x,y) \in (E^*(S)[\![x,y]\!])^1$$

given by identifying the right hand side with (the degree one part of) $E^*(\mathrm{Pic}\times\mathrm{Pic})$ and setting $f(x,y)$ to be the image of the first Chern class $c_1(\mathscr{L})$ of the universal line bundle $\mathscr{L}$ on Pic under the map

$$\otimes^*\colon E^*(\mathrm{Pic}) \to E^*(\mathrm{Pic}\times\mathrm{Pic})$$

coming from the tensor product of line bundles. Further, note that $f_E(x,y)$ is set up such that for all $X \in \mathrm{Sm}_S$ and $\mathscr{L}, \mathscr{L}' \in \mathrm{Pic}(X)$, we have

$$c_1(\mathscr{L}\otimes\mathscr{L}') = f_E(c_1(\mathscr{L}), c_1(\mathscr{L}'))$$

in E-cohomology.

8.1 Formal group laws

In this section, R denotes a classical commutative ring.

Definition 8.4. A (commutative, one-dimensional) gormal group law over R is a power series $f(x,y) \in R[\![x,y]\!]$ such that

1. $f(x,y) = f(y,x)$,
2. $f(x,0) = x = f(0,x)$, and
3. $f(x, f(y,z)) = f(f(x,y),z)$.

Remark 8.5. The unitality condition immediately implies that $f(x,y) = x+y+xy(\cdots)$ is given by the addition up to order one.

Definition 8.6. A morphism of formal group laws $\phi\colon f \to g$ is a power series $\phi(t) \in tR[\![t]\!]$ with no constant term such that

$$\phi(f(x,y)) = g(\phi(x),\phi(y)).$$

We see that ϕ as above is an isomorphism if the linear coefficient $\phi'(0) \in R^\times$ is a unit. An isomorphism is strict if $\phi'(0) = 1$.

Definition 8.7. To any commutative ring R, we associate the sets

$$\mathrm{FGL}(R), \qquad \mathrm{SI}(R)$$

of formal group laws and strict isomorphisms over R. The second is defined to be the set of triples $(f, g\phi)$ of two formal group law over R and a strict isomorphism $\phi\colon f \to g$.

Example 8.8. Not many formal group laws can be written out explicitly, but there are three elementary examples.

1. The additive formal group law f_a over any ring is defined by

$$f_a(x, y) = x + y.$$

2. The multiplicative formal group law f_m over any ring with a unit $u \in R^\times$ is defined by

$$f_m(x, y) = x + y + uxy.$$

3. Over any ring, one can also consider the formal group law given by

$$f(x, y) = \frac{x + y}{1 + xy}$$

   by noting that $1 + xy$ is a unit in $R[\![x, y]\!]$.

Further, let us notice that if R is a $\mathbb{Q}$-algebra, we can define the power series $\log(1 + ut) \in R[\![t]\!]$, and this defines a strict isomorphism between f_a and f_m over R. Similarly, if R is a $\mathbb{Z}_{(2)}$-algebra, there is a strict isomorphism between f and f_a given by $\tanh^{-1}(t)$.

Definition 8.9. If R is a $\mathbb{Z}$-graded ring, i.e. a commutative monoid object in $\mathrm{Ab}^{\mathbb{Z}}$ with Day convolution, then a graded formal group law over R is an element $f \in (R[\![x, y]\!])_{-1}$ satisfying the same axioms as above, where we set x, y to have degree -1.

Remark 8.10. Note that any formal group law can be written as

$$f(x, y) = \sum_{i,j \geq 0} a_{ij} x^i y^j.$$

Requiring that f be graded then means that a_{ij} should be in degree $i + j - 1$.

Remark 8.11. If $E \in \mathrm{CAlg}(\mathrm{hMS}_S)$ is oriented, $E^*(S)$ defined a graded ring, since C_2 acts trivially on $E^2(S) = E^0(\mathbb{P}^1 \otimes \mathbb{P}^1 \otimes S)$ by the naturality of the Thom isomorphism. A priori we could only view this as a Picard-graded object.

Definition 8.12. The Lazard ring L is defined to be

$$L := \mathbb{Z}[a_{ij} i, j \geq 0]/Q,$$

where Q is the ideal generated by the three relations that $a_i j$ should satisfy to define a formal group law as in Definition 8.4.

The Lazard ring then clearly carries a universal formal group law given by

$$f_{\mathrm{univ}}(x, y) = \sum_{i,j \geq 0} a_{ij} x^i y^j.$$

furthermore, it is clear that per construction there is a bijection

$$\mathrm{Ring}(L, R) \cong \mathrm{FGL}(R)$$

for any commutative ring R. We will now see that (strict) isomorphisms are also corepresentable.

Definition 8.13. Define the ring LB_+ by

$$LB_+ := L[b_0^{\pm}, b_1, b_2, \ldots],$$

and LB by

$$LB := LB_+/(b_0 - 1).$$

It is clear that LB_+ carries the universal isomorphism of formal group laws given by

$$\phi_{\mathrm{univ}}(t) = \sum_{i \geq 0} b_i t^i,$$

while LB carries the universal strict isomorphism of formal group laws. In both cases, the second formal group law is just the image of the universal one under this isomorphism, i.e. $g = \phi_{\mathrm{univ}}^{-1} f_{\mathrm{univ}}(\phi_{\mathrm{univ}}(x), \phi_{\mathrm{univ}}(y))$.

Remark 8.14. The functors FGL and SI form a groupoid object in $\mathrm{Fun}(\mathrm{CRing}, \mathcal{S}\mathrm{et})$ of the form

$$\mathrm{FGL} \leftleftarrows \mathrm{SI} \leftleftarrows \mathrm{SI} \times_{\mathrm{FGL}} \mathrm{SI} \qquad \cdots$$

with the simplicial structure maps given by projections of the form $(f, g, \phi) \mapsto f$ and degeneracies of the form $f \mapsto (f, f, \mathrm{id})$. Since all functors are corepresentable, we equivalently obtain a cogroupoid object in CRing of the form

$$L \rightrightarrows LB \rightrightarrows LB \otimes_L LB \qquad \cdots.$$

Since CRing is a 1-category, we see that this information is entirely encoded by the first three stages, i.e.

- the left and right units $\eta_L, \eta_R \colon L \to LB$,
- the augmentation map $\epsilon \colon LB \to L$, and
- the comultiplication map $\Delta \colon LB \to LB \otimes_L LB$,

satisfying the appropriate coherence relations. This structure is called a Hopf algebroid.

Remark 8.15. Given a unit $\lambda \in R^\times$ and a formal group law f over R, we can obtain a new formal group law f^λ by

$$f^\lambda(x, y) = \lambda f(\lambda^{-1}x, \lambda^{-1}y),$$

and this induces an action of $\mathbb{G}_m$ on FGL (by non-strict isomorphisms). Similarly, we obtain an action of $\mathbb{G}_m$ on SI by

$$\lambda \cdot (f, g, \phi) = (f^\lambda, g^\lambda, \lambda\phi(\lambda^{-1}-)).$$

These two actions are compatible with the groupoid structure maps, so we conclude that (L, LB) admits the structure of a graded Hopf algebroid.

Definition 8.16. Given a Hopf algebroid (A, Γ), define the category of (A, Γ)-comodules[3] by the totalisation

$$\mathrm{Mod}_{(A,\Gamma)} = \lim_{\Delta}\big(\mathrm{Mod}_A \rightrightarrows \mathrm{Mod}_\Gamma \rightrightarrows \mathrm{Mod}_{\Gamma \otimes_A \Gamma} \qquad \cdots \big)$$

And similarly in the graded case.

Remark 8.17. Since this totalisation is computed in 1-categories, we can be explicit and see that an (A, Γ)-comodule is an A-module M with a left A-linear map $\psi \colon M \to \Gamma \otimes_A M$ satisfying certain compatibilities.

[3]confusingly denoted by $\mathrm{Mod}_{(A,\Gamma)}$

8.2 Formal groups

Definition 8.18. A formal line over a commutative ring R is a functor $L\colon \mathrm{CAlg}_R \to \mathcal{S}\mathrm{et}_*$ such that there exists an isomorphism $L \cong \mathrm{Spf}(\widehat{\mathrm{Sym}}_R(M))$ for some invertible R-module $M \in \mathrm{Pic}(R)$. Since this lands in pointed sets, there is a section

$$e\colon \mathrm{Spec}(R) \to L.$$

If we write $L = \mathrm{Spf}(A)$, then e is a closed embedding with associated ideal $I = \ker(A \to R)$ given by the augmentation ideal, and conormal sheaf deenoted

$$\omega_L = I/I^2,$$

which can be identified with the cotangent space of L at e.

Remark 8.19. Given an isomorphism as in the definition, we see that $\omega_L \cong M$ so ω_L is always invertible.

Definition 8.20. A (commutative, one-dimensional, smooth) formal group over R is a functor

$$G\colon \mathrm{CAlg}_R \to \mathrm{Ab}$$

such that the composite the forgetful functor

$$\mathrm{CAlg}_R \xrightarrow{G} \mathrm{Ab} \to \mathcal{S}\mathrm{et}_*$$

defines a formal line over R.

Remark 8.21. From the definition, we see that a formal group over R is precisely a group structure on a formal line over R. In fact, we can rephrase the definition of a formal group law as the datum of a group structure on the formal affine line $\mathrm{Spf}\, R[\![x]\!] = \widehat{\mathbb{A}}^1_R$.

Remark 8.22. Let L be a formal line over R, and let $M \in \mathrm{Pic}(R)$ be the invertible module such that $L \cong \mathrm{Spf}(\widehat{\mathrm{Sym}}_R(M))$, then Zariski locally on $\mathrm{Spec}(R)$, M is free, whence L is Zariski locally of the form $\widehat{\mathbb{A}}^1_R$ and we see that any formal group is Zariski-locally equivalent to a formal group law.

Definition 8.23. Define the moduli of formal groups to be the functor

$$\mathscr{M}_{\mathrm{fg}}\colon \mathrm{CRing} \to \mathrm{Gpd}$$

that sends a ring R to the groupoid of formal groups over R and isomorphisms between these.

Remark 8.24. We will construct a presentation of this stack, but the rings involved are not finitely generated. In fact, $\mathscr{M}_{\mathrm{fg}}$ is not an algebraic stack.

There exists a natural map

$$\mathscr{M}_{\mathrm{fg}} \to \mathrm{Pic}, G \mapsto \omega_G,$$

and identifying Pic with $B\mathbb{G}_m$, we see that this defines a line bundle on $\mathscr{M}_{\mathrm{fg}}$. Let $\mathscr{M}^s_{\mathrm{fg}} \to \mathscr{M}_{\mathrm{fg}}$ denote the corresponding $\mathbb{G}_m$-torsor. The per construction, $\mathscr{M}^s_{\mathrm{fg}}$ classifies pairs of a formal group G over R and a trivialisation $\omega_G \cong R$.

Proposition 8.25. *Let R be a commutative ring, then there is an $R^\times$-equivariant equivalence*

$$\mathscr{M}^s_{\mathrm{fg}} \cong \operatorname*{colim}_{\Delta^{\mathrm{op}}}(\ \mathrm{FGL}(R) \leftleftarrows \mathrm{SI}(R) \mathrel{\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow}} \cdots\ )$$

More precisely, $\mathscr{M}^s_{\mathrm{fg}}$ is presented by the graded Hopf algebroid (L, LB) as a $\mathbb{G}_m$-stack.

Proof. Let G be an R-point of $\mathscr{M}_{\mathrm{fg}}^s$, i.e. a formal group $G \cong \mathrm{Spf}(A)$ with a trivialisation $\omega_G = I/I^2 \cong R$. Then the augmentation ideal $I \subset A$ clearly surjects onto ω_G per construction, and we let $t \in A$ denote a lift of $1 \in R$. This induces a continuous map

$$R[\![t]\!] \to A,$$

and we see that on associated graded this recovers the map

$$I^n/I^{n+1} \to \mathrm{Sym}_R^n(\omega_G)$$

which is an isomorphism by the trivialisation. We conclude that the map above is an isomorphism. Therefore, the map

$$\mathrm{FGL}(R) \to \mathscr{M}_{\mathrm{fg}}^s(R)$$

is an effective epimorphism (i.e. a surjection on π_0). It suffices to note that

$$\mathrm{SI}(R) \cong \mathrm{FGL}(R) \times_{\mathscr{M}_{\mathrm{fg}}^s(R)} \mathrm{FGL}(R)$$

from the definition, so that the cosimplicial diagram in the statement of the proposition is the Čech nerve of this effective epimorphism, and we are done. □

Remark 8.26. More precisely, note that there is a pullback diagram

$$\begin{tikzcd}
\mathrm{Spec}(L) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathscr{M}_{\mathrm{fg}}^s \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & * \arrow[d] \\
\mathrm{Spec}(L)/\mathbb{G}_m \arrow[r] & \mathscr{M}_{\mathrm{fg}} \arrow[r] & \mathrm{B}\mathbb{G}_m,
\end{tikzcd}$$

and we are claiming that the bottom left horizontal map is a faithfully flat and affine. This follows since the left unit $\eta_L\colon L \to LB$ is the inclusion of the coefficients into a polynomial ring hence clearly faithfully flat. By the axioms of the Hopf algebroid, so is η_R.

Remark 8.27. In particular, the presentation above gives a cosimplicial model for $\mathrm{QCoh}(\mathscr{M}_{\mathrm{fg}})$, and we see that

$$\mathrm{QCoh}(\mathscr{M}_{\mathrm{fg}}) \cong \mathrm{Mod}_{(L,LB)}^{\mathrm{gr}} \cong \mathrm{QCoh}^{\mathbb{G}_m}(\mathscr{M}_{\mathrm{fg}}^s).$$

The following theorem gives a useful criterion for checking when a map to $\mathscr{M}_{\mathrm{fg}}$ classifying a formal group law over a ring is flat.

Theorem 8.28 (Landweber). *Let $f \in \mathrm{FGL}(R)$ be a formal group law over a commutative ring R, classified by a map*

$$\mathrm{Spec}(R) \to \mathrm{Spec}(L) \to \mathscr{M}_{\mathrm{fg}}.$$

Then the composite map above is flat if and only if for all primes p, the sequence $(v_0 = p, v_1, v_2, \ldots)$ is regular in R, where v_i is the coefficient of x^{p^i} in the p-series $[p]_f(x) \in R[\![x]\!]$.

Remark 8.29. Recall that the n-series of a formal group law f over R is defined inductively by $[1]_f(x) = x$ and $[n]_f(x) = f(x, [n-1]_f(x))$. In particular, this always has leading terms nx.

Example 8.30. Let R be $\mathbb{Z}[\beta^{\pm}], |\beta| = 1$ and consider the multiplicative formal group law over R with unit $u = -\beta$, i.e. $f_m(x,y) = x + y + -\beta xy$. Then f_m is flat, since the formula

$$x + y - \beta xy = \beta^{-1}(1 - (1-\beta x)(1-\beta y))$$

shows us that the p-series is given by

$$[p]_{f_m}(x) = \beta^{-1}(1 - (1-\beta x)^p),$$

so that $v_0 = p$, $v_1 = (-\beta)^{p-1}$, and all higher v_i's are zero. This clearly forms a regular sequence in R.

Example 8.31. The additive formal group law over a ring R is not flat unless R is a $\mathbb{Q}$-algebra. Indeed $[p]_{f_a}(x) = px$.

8.3 Algebraic cobordism and its Hopf algebroid

Proposition 8.32. *If $E \in \mathrm{CAlg}(\mathrm{hMS}_S)$ is oriented, then*

$$E \otimes \mathrm{MGL} \cong E[b_1, b_2, \ldots]$$

in $\mathrm{CAlg}(\mathrm{hMS}_S)$*. Moreover, let $c_E \in E^1(\mathrm{Pic})$ be the first Chern class of the universal line bundle, then the two classes c_E and c_{MGL} in $(E \otimes \mathrm{MGL})^1(\mathrm{Pic})$ are related by*

$$c_{\mathrm{MGL}} = \sum_{i \geq 0} b_i c_E^{i+1}. \tag{2}$$

This expression for $E_*\mathrm{MGL}$ follows directly from our computations of $M^*\mathrm{MGL}$ for any E-module M. Indeed, it is clear that $[X, E \otimes \mathrm{MGL}] \cong [X \otimes \mathrm{MGL}, E \otimes \mathrm{MGL}]_{\mathrm{MGL}}$.

Remark 8.33. Note that the classes b_i above arise from the identification

$$E_*(\mathrm{Pic}) \cong (E^*(\mathrm{Pic}))^\vee \cong \bigoplus_{n \geq 0} E_*\{\beta_n\},$$

where we use our computaiton $E^*(\mathrm{Pic}) \cong E^*[\![c_1]\!]$, and $\{\beta_n\}_n$ is a dual basis to $\{c_1^n\}_n$. Now there is a map

$$E_*(\mathrm{Pic}) \to E_*(\mathrm{Vect}_\infty) \cong E_*[\beta_1, \beta_2, \ldots],$$

where we used that Vect_∞ becomes a ring object in MS_S to obtain an isomorphism of rings. The Thom isomorphism identifies

$$E_*(\mathrm{Vect}_\infty) \cong E_*(\mathrm{MGL}),$$

and β_i get sent to the class we call b_i on the right hand side. Beware that we also computed

$$E^*(\mathrm{Vect}_\infty) \cong E^*[\![c_1, c_2, \ldots]\!],$$

but the classes β_i are not dual to c_i.

Corollary 8.34. *If S is a qcqs base, then the homotopy groups of the Čech conerve*[4] *of the unit map $\mathbb{1} \to \mathrm{MGL}$ give rise to a Hopf algebroid* $(\mathrm{MGL}_*, \mathrm{MGL}_*\mathrm{MGL})$.

Remark 8.35. This is a sort of Adams type or Ravenel flat condition on MGL, indeed setting $E = \mathrm{MGL}$ in the proposition shows that $\mathrm{MGL}_*\mathrm{MGL}$ splits as a sum of copies of MGL_*, so that the comultiplication just comes from a codegeneracy in the Amitsur complex by the equivalences

$$\Delta\colon \mathrm{MGL}_*\mathrm{MGL} \cong (\mathrm{MGL} \otimes \mathrm{MGL})_* \to (\mathrm{MGL} \otimes \mathrm{MGL} \otimes \mathrm{MGL})_* \cong \mathrm{MGL}_*\mathrm{MGL} \otimes_{\mathrm{MGL}_*} \mathrm{MGL}_*\mathrm{MGL}.$$

Remark 8.36. If $X \in \mathrm{MS}_S$ is any motivic spectrum, MGL_*X admits the structure of a $(\mathrm{MGL}_*, \mathrm{MGL}_*\mathrm{MGL})$-comodule, and if $X \in \mathrm{MS}_S^\omega$ is a compact motivic spectrum, we can dualise to see that MGL^*X also admits such a comodule structure.

Now note that (2) with $E = \mathrm{MGL}$ implies that there is a commutative diagram

$$\begin{tikzcd} L \arrow[r] \arrow[d, "\eta_L"] & \mathrm{MGL}_* \arrow[d, "\eta_L"] \\ LB \arrow[r] & \mathrm{MGL}_*\mathrm{MGL}, \end{tikzcd}$$

which is in fact a coCartesian square of graded commutative rings, where the bottom arrow sends b_i to b_i. Therefore, $(\mathrm{MGL}_*, \mathrm{MGL}_*\mathrm{MGL})$-comodules are base-changed in the sense that

$$\mathrm{Mod}^{\mathrm{gr}}_{(\mathrm{MGL}_*, \mathrm{MGL}_*\mathrm{MGL})} \simeq \mathrm{Mod}^{\mathrm{gr}}_{(L, LB)} \times_{\mathrm{Mod}^{\mathrm{gr}}_L} \mathrm{Mod}^{\mathrm{gr}}_{\mathrm{MGL}_*},$$

i.e. an $(\mathrm{MGL}_*, \mathrm{MGL}_*\mathrm{MGL})$-comodule is an MGL_*-module with an (L, LB)-comodule structure.

[4]a.k.a. Amitsur complex

8.3.1 Cohomology theories and the Conner–Floyd isomorphism

Let S be a qcqs base, and consider the category

$$\mathrm{CohThy}(\mathrm{MS}_S^{\omega}) \subset \mathrm{Fun}(\mathrm{MS}_S^{\omega,\mathrm{op}}, \mathrm{Ab})$$

of cohomology theories on compact motivic spectra, i.e. the full subcategory on functors that send cofibre sequences to exact sequences and preserve finite products. A generic object in here is denoted E^0 (and it is clear that every motivic spectrum defines a cohomology theory on MS_S^{ω}), and we write E^q for $E^0 \circ \Sigma_{\mathbb{P}^1}^{-q}$, $E^{p,q}$ for $E^q \circ \Sigma^{2q-p}$. A ring cohomology theory is a commutative monoid in $\mathrm{CohThy}(\mathrm{MS}_S^{\omega})$ for the Day convolution symmetric monoidal structure.

Remark 8.37. Note that $E^{p,q}$ is only a functor $\mathrm{MS}_S^{\omega,\mathrm{op}} \to \mathrm{Ab}$. We can however extend this to the entire category by taking Ind (up to some op's) to obtain a functor denoted

$$\widehat{E}^{p,q} \colon \mathrm{MS}_S^{\mathrm{op}} \to \mathrm{Pro}(\mathrm{Ab})$$

Let us single out certain ring cohomology theories of interest.

- E^0 is said to be periodic if we are additionally given a unit $\beta \in E^{-1}(\mathbb{1})$.
- E^0 is said to be oriented if we are additionally given a class $c_1 \in \widehat{E}^1(\mathrm{Pic})$ such that restriction along the map $\mathbb{P}^1 \to \mathrm{Pic}$ sends this class to $(0,1) \in E^1(\mathbb{P}^1) \cong E^1(\mathbb{1}) \oplus E^0(\mathbb{1})$
- E^0 is said to be $\mathbb{G}_m$-pre-oriented if we are given a class $u \in \widehat{E}^0(\mathrm{Pic})$ such that restriction along the map $\mathbb{1} \to \mathrm{Pic}$ sends this to $1 \in E^0(\mathbb{1})$, and restriction along the map $\otimes\colon \mathrm{Pic} \times \mathrm{Pic} \to \mathrm{Pic}$ sends u to the product $u_1 u_2$, where $u_i = \pi_i^* u \in \widehat{E}^0(\mathrm{Pic} \times \mathrm{Pic})$. In this case, define

$$\beta \coloneqq 1 - u\mid_{\mathbb{P}^1} \in E^0(\mathbb{P}^1) = E^{-1}(\mathbb{1}).$$

- E^0 is said to be $\mathbb{G}_m$-oriented if it is $\mathbb{G}_m$-pre-oriented and the class β defined above is a unit.

Now note that by Landweber's criterion (Theorem 8.28), we saw that the multiplicative formal group law over $\mathbb{Z}[\beta^{\pm}]$ with β in degree one was flat over $\mathscr{M}_{\mathrm{fg}}$. Therefore, we see that the functor

$$\mathrm{MGL}^*(-) \otimes_L \mathbb{Z}[\beta^{\pm}] \colon \mathrm{MS}_S^{\omega,\mathrm{op}} \to \mathrm{Ab}$$

still defines a cohomology theory on MS_S^{ω}, since it still preserves finite products and exact sequences by flatness. In fact, direct verification shows that this is the initial oriented ring cohomology theory with formal group law given by $x + y - \beta x y$.

On the other hand $\mathrm{KGL}^*(-)$ has a universal property as a cohomology theory directly inherited from its universal property as a motivic spectrum. Indeed, we saw that

$$\mathrm{KGL} \simeq \Sigma_{\mathbb{P}^1}^{\infty} \mathrm{Pic}_+[\beta^{-1}],$$

so that $\mathrm{KGL}^*(-)$ is the initial $\mathbb{G}_m$-oriented ring cohomology theory.

The claim is now that these conditions, namely being $\mathbb{G}_m$-oriented and oriented with multiplicative formal group law, are equivalent, whence the desired isomorphism of ring cohomology theories follows.

9 Motivic filtrations

In this section, we discuss how to obtain filtrations on motivic spectra representing localising invariants (or more generally) coming from previously established filtrations.

9.1 Motivic spectra from $\mathbb{E}_\infty$-rings

Let $\mathscr{C} \in \mathrm{CAlg}(\mathcal{C}\mathrm{at}_\infty)$, then note that there are symmetric monoidal functors

$$\mathrm{Fin}^{\simeq} \to \mathbb{N} \to \mathbb{Z}_{\leq}^{\mathrm{op}} \to *$$

given by taking the cardinality, inclusion, and projection respectively. Since these are symmetric monoidal they induce lax symmetric monoidal functors on functor categories equipped with Day convolution of the form

$$\mathscr{C} \to \mathrm{Fil}(\mathscr{C}) \to \mathscr{C}^{\mathbb{N}} \to \mathrm{SSeq}(\mathscr{C}).$$

There are to be interpreted as equipping an object with the constant filtration, forgetting negative filtrants and structure maps, and inserting trivial Σ_n-actions respectively.

Definition 9.1. Let $F^\star E$ be an object of $\mathrm{CAlg}(\mathrm{Fil}(\mathcal{S}\mathrm{hv}(\mathrm{Sm}_S;\mathrm{Sp})_{\mathrm{Nis}})) =: \mathrm{Fil}(\mathscr{V})$. Then an orientation of $F^\star E$ is a map

$$c\colon \Sigma^\infty \mathrm{Pic} \to F^1 E$$

such that the filtered projective bundle formula holds, i.e. for all $X \in \mathrm{Sm}_S$, $r \geq 1$ and $n \in \mathbb{Z}$, the map

$$\sum_{i=0}^{r} c(\mathscr{O}(1))^i \colon \bigoplus_{i=0}^{r} F^{n-i}E(X) \to F^n E(\mathbb{P}^r_X)$$

is an equivalence.

Proposition 9.2. *Let $(F^\star E, c)$ be as in the definition above. Then we can construct an $\mathbb{E}_\infty$-ring $e \in \mathrm{CAlg}(\mathrm{MS}_S)$ with a Bott element $\beta\colon \mathbb{P}^1 \to e$ such that the following hold.*

- *For every n we can identify*
$$\Omega^{\infty-n}_{\mathbb{P}^1} e \simeq F^n E.$$
This is an equivalence of $\mathbb{E}_\infty$-rings if $n = 0$
- *For every n there is an identification of maps*
$$\Omega^{\infty-n}_{\mathbb{P}^1}(\Sigma_{\mathbb{P}^1} e \xrightarrow{\beta} e) \simeq F^{n+1}E \to F^n E.$$
- *There is an equivalence of $\mathbb{E}_\infty$-rings*
$$\Omega^{\infty}_{\mathbb{P}^1} e \simeq F^{-\infty} E.$$
- *The quotient e/β inherits an $\mathbb{E}_\infty$-ring structure such that*
$$\Omega^{\infty-n}_{\mathbb{P}^1} \simeq \mathrm{gr}_n E,$$
and this is an equivalence of $\mathbb{E}_\infty$-rings if $n = 0$.

Proof. First, note that restricting c along $\mathbb{P}^1 \to \mathrm{Pic}$, we obtain a map

$$\begin{tikzcd} \mathbb{P}^1 \arrow[r, "c|_{\mathbb{P}^1}"] \arrow[dr, "\beta"'] & F^1 E \arrow[d] \\ & F^0 E. \end{tikzcd}$$

This can be viewed as an $\mathbb{E}_\infty$-map between $\{0 \le 1\}^{\mathrm{op}}$-filtered algebras, so after applying the lax symmetric monoidal functor to symmetric sequences we obtain an $\mathbb{E}_\infty$ ring map

$$\mathrm{Free}_{\mathbb{E}_\infty}(0, \mathbb{P}^1, 0, 0, \ldots) \to (F^0E, F^1E, \ldots),$$

exhibiting the right hand side as an $\mathbb{E}_\infty$ algebra e in

$$\mathrm{Mod}(\mathrm{SSeq}(\mathscr{V}); \mathrm{Free}_{\mathbb{E}_\infty}(0, \mathbb{P}^1, 0, 0, \ldots)) = \mathrm{Sp}^{\mathrm{lax}}_{\mathbb{P}^1}(\mathscr{V}).$$

The map β then induces a map $\beta\colon \mathbb{P}^1 \to \Omega^\infty_{\mathbb{P}^1} e$ per construction. Next, we note that the filtered PBF with $r = 1$ implies that e is a strict $\mathbb{P}^1$-spectrum, and the filtered PBF at $r \ge 2$ implies that it satisfies elementary blowup excision. Together, these exhibit

$$e \in \mathrm{CAlg}(\mathrm{MS}_S),$$

and it is clear that its n-th space has the description above. It then suffices to note that the span

$$F^{-\infty}E \leftarrow F^{\star}E \to \mathrm{gr}_*E$$

can be viewed as a span of $\mathbb{E}_\infty$-algebras in $\mathrm{Fil}(\mathscr{V})$, where we endow the object on the left with the constant filtration and the object on the right with zero maps[5]. Then functoriality of the construction above gives us the desired span

$$e[\beta^{-1}] \leftarrow e \to e/\beta$$

in $\mathrm{CAlg}(\mathrm{MS}_S)$. $\square$

This construction is already interesting even when the filtration is constant. Let us discuss two examples.

Example 9.3. Let

$$E\colon \mathfrak{Cat}^{\mathrm{st}}_\infty \to \mathrm{Sp}$$

be a localising invariant. If S is a qcqs scheme, we view this as a functor

$$E\colon \mathrm{Sm}_S^{\mathrm{op}} \xrightarrow{\mathrm{Perf}(-)} \mathfrak{Cat}^{\mathrm{st}}_\infty \xrightarrow{E} \mathrm{Sp}.$$

Since it is a localising invariant, it satisfies Zariski and in fact Nisnevich descent, and by the universality of K-theory, it is a module over (nonconnective) K-theory[6]. There is an orientation on K defined by

$$c\colon \Sigma^\infty \mathrm{Pic} \to \mathrm{K}$$

classifying $\mathscr{O} - \mathscr{L}^\vee_{\mathrm{univ}}$. One can then verify that E satisfies the (trivially filtered) PBF with respect to the orientation it inherits from K. Therefore, the construction above gives us

$$E_S \in \mathrm{Mod}(\mathrm{MS}_S; \mathrm{KGL}).$$

If E was multiplicative, the same construction gives us an object in $\mathrm{CAlg}(\mathrm{MS}_S)_{\mathrm{KGL}/}$.

Example 9.4. Let (A, I) be a prism and S a qcqs scheme with a map to $\mathrm{Spec}(A/I)$. Denote

$$(\Delta_{-/A}\{n\}[2n])_{n \in \mathbb{Z}} \in \mathrm{CAlg}(\mathrm{Sp}^{\mathrm{Gr}}) \to \mathrm{CAlg}(\mathrm{Sp}^{\mathrm{Fil}})$$

the Breuil–Kisin twisted prismatic cohomology sheaf on Sm_S. The construction above gives us

$$\mathrm{H}A^{\Delta} \in \mathrm{CAlg}(\mathrm{MS}_S)$$

which represents prismatic cohomology over A, as can be easily seen from the description of its n-th spaces as a $\mathbb{P}^1$-spectrum. In the special case where $A = W(k), I = (p)$ for k a perfect field of characteristic p, this recovers crystalline cohomology.

[5]For this it suffices to note that the inclusion $\mathbb{Z}^{\delta} \to \mathbb{Z}^{\mathrm{op}}_{\le}$ is symmetric monoidal and right Kan extension along this inclusion in $\mathrm{Fun}(-, \mathscr{V})$ is given explicitly by adding zero structure maps and lax symmetric monoidal since it is right adjoint to a strong symmetric monoidal functor.

[6]Simply denoted K in this section.

9.2 The motivic filtration on algebraic K-theory

Let us first recall the desiderata for a motivic filtration on algebraic K-theory and its historical context. It should be a complete $\mathbb{N}_{\leq}^{\mathrm{op}}$-indexed filtration $F^\star\mathrm{K}$ such that $F^0\mathrm{K} = \mathrm{K}$ (hence trivially exhaustive), where the first few stages can be described explicitly.

- Let $\mathrm{rk}\colon \mathrm{K} \to \mathbb{Z}$ be the rank map. Then we want
$$F^1 K = \mathrm{fib}(\mathrm{K} \to \mathbb{Z}).$$
- Let $\det\colon \mathrm{K} \to \mathrm{Pic}^{\mathbb{Z}}$ be the determinant map landing in an extension of Pic, i.e. $\mathrm{Pic}^{\mathbb{Z}}(X) = \mathrm{Pic}(\mathrm{Perf}(X)) = \{\mathscr{L}[n] \mid \mathscr{L} \in \mathrm{Pic}(X), n \in \mathbb{Z}$. Then we want
$$F^2\mathrm{K} = \mathrm{fib}(\mathrm{K} \to \mathrm{Pic}^{\mathbb{Z}}).$$

In particular, we obtain $\mathrm{gr}_0\mathrm{K} = \mathbb{Z}$ and $\mathrm{gr}_1\mathrm{K} = \mathrm{Pic}$. More generally, let us write $\mathrm{gr}_n\mathrm{K} = \mathbb{Z}(n)[2n]$.

Remark 9.5. The motivic filtration on K-theory is constructed as a resolution to Beilinson's conjectures from the '80s. In particular, he predicted that the filtration should split rationally, with associated graded given by the Adams eigenspaces on K-theory, and that there should be an étale comparison theorem for ℓ-adic motivic cohomology of schemes where ℓ is invertible.

Let us now discuss some of the history behind various constructions and refinements of motivic filtrations on algebraic K-theory.

- Bloch in '86 defined the cycle complex in analogy with singular cohomology, as a candidate for motivic cohomology.
- Bloch–Lichtenbaum in '94 showed that this constructs a motivic filtration on the K-theory of a field.
- Friedlander–Suslin in '00 as well as Grayson around the same time extended this to a motivic filtration on K-theory of smooth schemes over a field.
- Voevodsky in '02 developed the slice filtration and showed that for a smooth scheme over a characteristic zero field its associated graded recovers Bloch's cycle complex. The upshot of the slice filtration is that it provided a manifestly multiplicative filtration.
- Levine in '05 extended Voevodsky's comparison to smooth schemes over characteristic p fields and Dedekind domains, but it is not apparent fro his construction that the resulting filtration is multiplicative.
- Spitzweck in '12 shows that there is a multiplicative structure on the associated graded (but not the filtration itself).
- Bachmann in '22 proved that the slice filtration works over general Dedekind domains, hence showing that the corresponding filtraition is multiplicative.
- Bachmann–Elmanto–Morrow in '24 define an extension of the motivic filtration and motivic cohomology to singular schemes, hence approximating their K-theory as opposed to their homotopy K-theory.
- Elmanto–Morrow in '23 and Bouis in '24 extend this further.

The results all the way up to Bachmann's in '22 can be synthesis in the theorem below.

Theorem 9.6. *Let S be a Dedekind scheme, then there exists a complete multiplicative $\mathbb{N}_{\leq}^{\mathrm{op}}$-indexed filtration $F^\star K$ in $\mathscr{V} = \mathcal{S}\mathrm{hv}(\mathrm{Sm}_S;\mathrm{Sp})_{\mathrm{Nis},\mathbb{A}^1}$ such that if we denote $\mathrm{gr}_n\mathrm{K}$ by $\mathbb{Z}(n)[2n]$ then $\mathbb{Z}(n)[2n] \simeq |z^n(-;\bullet)|$.*

Remark 9.7. Recall that $z^n(X,\bullet)$ is Bloch's cycle complex, defined as the simplicial abelian group that in degree k is the free abelian group on irreducible codimension k subschemes of $X \times \Delta^k$ that intersect all faces properly.

Plugging this theorem into the machine described in Section 1 then gives us a motivic spectrum $\mathrm{kgl} \in \mathrm{CAlg}(\mathrm{MS}_S)$ with a Bott element $\beta\colon \mathbb{P}^1 \to \mathrm{kgl}$ such that

- $\mathrm{kgl}[\beta^{-1}] \simeq \mathrm{KGL}$,
- $\mathrm{kgl}/\beta \simeq \mathrm{H}\mathbb{Z}$, and
- $\Omega^{\infty-n}_{\mathbb{P}^1}\mathrm{kgl} \simeq F^n K$.

9.3 Motivic filtrations on TC &c.

Recall that THH is a localising invariant

$$\mathrm{THH}\colon \mathcal{C}\mathrm{at}^{\mathrm{st}}_{\infty} \to \mathrm{CycSp} \to \mathrm{Sp}^{\mathrm{B}\mathbb{T}}.$$

From it, we produce

$$\mathrm{TC}^- = \mathrm{THH}^{h\mathbb{T}}, \qquad \mathrm{TP} = \mathrm{THH}^{t\mathbb{T}}, \qquad \mathrm{TC}^+ = \mathrm{THH}_{h\mathbb{T}},$$

where the first two are still localising invariants. These are related by a cofibre sequence

$$\Sigma\mathrm{TC}^+ \xrightarrow{\mathrm{Nm}} \mathrm{TC}^- \to \mathrm{TP}.$$

Topological cyclic homology is then defined as the global sections

$$\mathrm{TC} = \mathrm{Map}_{\mathrm{CycSp}}(\mathbb{1}, \mathrm{THH}).$$

Remark 9.8. If the underlying spectrum of $\mathrm{THH}(\mathscr{C})$ is bounded below, for example if $\mathscr{C} = \mathrm{Perf}(X)$, then there is a more explicit equaliser formula due to Nikolaus–Scholze:

$$\mathrm{TC}(\mathscr{C}) \longrightarrow \mathrm{TC}^-(\mathscr{C}) \underset{\mathrm{can}}{\overset{\phi}{\rightrightarrows}} \mathrm{TP}^\wedge,$$

where TP is now profinitely completed.

Remark 9.9. Further, recall from the general theory of localising invariants that there is a cyclotomic trace map $\mathrm{tr}\colon \mathrm{K} \to \mathrm{TC}$, that all localising invariants (so everything except TC^+) mentioned above have $\mathbb{E}_\infty$ structures, and that all are oriented, satisfying the filtered PBF for the constant filtration.

From now on, we implicitly fix a prime p, and TC and its cousins are all implicitly p-completed. Let E denote any of $\{\mathrm{TC}, \mathrm{THH}, \mathrm{TP}, \mathrm{TC}^-\}$.

9.4 The BMS filtration

Bhatt–Morrow–Scholze defined a motivic filtration on E, i.e. a refinement

$$F^\star E\colon \mathrm{dSch}^{\mathrm{op}}_{\mathrm{qcqs}} \to \mathrm{CAlg}(\mathrm{Sp}^{\mathrm{Fil}}).$$

These filtrations can in fact be relatively well described.

- The filtration $F^\star E$ is commplete.
- The filtrations on THH and TC are furthermore exhaustive since they are in fact constant in negative degrees.

- The filtrations on TC^- and TP are exhaustive on quasisyntomic schemes.
- There are identifications of the associated graded

$$\mathrm{gr}_n\mathrm{TP} \simeq \widehat{\mathbb{\Delta}}\{n\}[2n], \quad \mathrm{gr}_n\mathrm{TC}^- \simeq \mathrm{Fil}^n_{\mathrm{N}}\widehat{\mathbb{\Delta}}\{n\}[2n], \quad \mathrm{gr}_n\mathrm{THH} \simeq \mathrm{gr}^n_{\mathrm{N}}\mathbb{\Delta}\{n\}[2n], \quad \mathrm{gr}_n\mathrm{TC} \simeq \mathbb{Z}_p^{\widehat{\mathrm{syn}}}\{n\}[2n].$$

In the last identification, $\mathbb{Z}_p^{\widehat{\mathrm{syn}}}$ denoted the syntomic cohomology of $p$adic formal schemes, which when viewed as a functor on general schemes only depends on the p-completion. Later we will also encounter the global syntomic complexes which are obtained by gluing in the étale cohomology of the generic fibre.

Remark 9.10. With respect to the BMS filtration, the orientation of K factors uniquely as

$$\begin{tikzcd} \Sigma^\infty\mathrm{Pic} \arrow[r] \arrow[drr, dotted, "\exists!"'] & \mathrm{K} \arrow[r, "\mathrm{tr}"] & \mathrm{TC} \\ & & F^1\mathrm{TC} \arrow[u] \end{tikzcd}$$

In fact, the same can be said for other choices of E, since these are related by maps

$$\begin{tikzcd} & & \mathrm{THH} \\ \mathrm{TC} \arrow[r] & \mathrm{TC}^- \arrow[ur] \arrow[dr] & \\ & & \mathrm{TP}. \end{tikzcd}$$

One verifies that with respect to this filtered orientation, E satisfies that filtered projective bundle formula. This is nontrivial, but since these filtrations are complete it suffices to check this on associated graded, where it is a result of Bhatt–Lurie[7].

Feeding this into the machine of Section 1, we therefore obtain $\mathbb{E}_\infty$-rings in MS_S with Bott elements, denoted

$$\begin{tikzcd} & & \mathrm{thh} \\ \mathrm{tc} \arrow[r] & \mathrm{tc}^- \arrow[ur] \arrow[dr] & \\ & & \mathrm{tp}. \end{tikzcd}$$

However, it is not clear that the filtrations are sufficiently compatible that the cylotomic trace induces a map $\mathrm{kgl} \to \mathrm{tc}$. This will require a more refined analysis of the BMS filtration.

Remark 9.11. Consider the inclusions

$$\mathrm{Poly}_{\mathbb{Z}} \hookrightarrow \mathrm{Sm}_{\mathbb{Z}} \xrightarrow{j} \mathrm{QSyn}_{\mathbb{Z}},$$

where the final category is equipped with the quasisyntomic topology. Then BMS prove the following.

- E is a sheaf for the quasisyntomic topology, and in fact left Kan extended from its values on $\mathrm{Poly}_{\mathbb{Z}}$.
- On $\mathrm{QSyn}_{\mathbb{Z}}$, the motivic filtration takes the form

$$F^nE = \tau^{\mathrm{qsyn}}_{\geq 2n}E,$$

  and in fact every filtration stage is also left Kan extended from its values on $\mathrm{Poly}_{\mathbb{Z}}$.

[7] The orientation that Bhatt–Lurie provide may not agree with the orientation inherited from algebraic K-theory by means of the trace map, so this is nontrivial to verify.

- The cofibre $\mathrm{TC}/F^n\mathrm{TC}$ is qsyn-locally $2n$-coconnective.

By these points, if we want a lift in the diagram

$$\begin{tikzcd}
F^n\mathrm{K} \arrow[r, dotted] \arrow[d] & F^n\mathrm{TC} \arrow[d] \\
\mathrm{K} \arrow[r, "\mathrm{tr}"] & \mathrm{TC} \arrow[d] \\
& \mathrm{TC}/F^n\mathrm{TC}
\end{tikzcd}$$

providing a filtered refinement of the cylotomic trace, it will suffice to show that the left Kan extension $j_! F^n\mathrm{K}$ is qsyn-locally $2n$-connective. In fact, this will not only give us a filtered refinement but it will formally lift to a multiplicative filtered refinement as well. Let us sketch the idea of how to prove this.

- Note that $F^\star\mathrm{K}$ is a complete filtration, so it suffices to check our connectivity desideratum on the associated graded.
- The complexes $\mathbb{Z}_p(n)[2n]$ on Sm_S are Zariski-locally n-connective by a result of Geisser.
- There is an equivalence due to Geisser and Bhatt–Mathew of the form

$$L_{\text{ét}}\mathbb{Z}_p(n)[2n] \simeq \mathbb{Z}_p^{\mathrm{syn}}(n)[2n].$$

- The complexes $\mathbb{Z}_p^{\mathrm{syn}}(n)[2n]$ are left Kan extended from their values on Sm_S (but not polynomial algebras).
- Furthermore, these are qsyn-locally $2n$-connective by Bhatt–Scholze.

10 Atiyah duality

We will prove a duality theorem for smooth projective schemes over S, i.e. by identifying their duals in MS_S with the Thom spectrum of (the dual of) their cotangent bundle.

Theorem 10.1. *Let $f\colon X \to S$ be a smooth projective morphism. Then there is a canonica equivalence*

$$f_\sharp \Sigma^{-\Omega_f} \xrightarrow{\sim} f_*$$

of functors $\mathrm{MS}_X \to \mathrm{MS}_S$.

Corollary 10.2. *We immediately see that for any smooth projective S-scheme X, $\Sigma^\infty_{\mathbb{P}^1} X_+$ is dualisable with dual given by $\mathrm{Th}_X(-\Omega_{X/S})$. Furthermore, this provides a notion of Poincaré duality for oriented cohomology theories with a Künneth formula.*

The proof of this theorem follows from a more general construction of Gysin maps.

10.1 Gysin maps

Consider a closed embedding $Z \hookrightarrow X$ in S.

Remark 10.3. in $\mathbb{A}^1$-invariant homotopy theory, we have the Morel–Voevodsky purity theorem, which establishes an equivalence

$$\frac{X}{X \setminus Z} \xrightarrow{\sim} \mathrm{Th}_Z(N_{Z/X}).$$

This is constructed by considering a zig-zag of the form

$$\frac{X}{X \setminus Z} \xrightarrow{i_1} \frac{D_{Z/X}}{D_{Z/X} \setminus (Z \times \mathbb{A}^1)} \xleftarrow{i_0} \frac{N_{Z/X}}{N_{Z/X} \setminus Z}.$$

The central term is the deformation to the normal cone, which is a map

$$\mathrm{Bl}_{Z\times 0}(X \times \mathbb{A}^1) \setminus \mathrm{Bl}_{Z\times 0}(X \times 0) = D_{Z/X} \xrightarrow{i} \mathbb{A}^1$$

with fibres $i_0 = N_{Z/X}$, $i_\lambda = X$ for any nonzero λ. One then applies the tubular neighbourhood theorem to see that this zig-zag induces an equivalence in $\mathrm{MS}_S^{\mathbb{A}^1}$.

In the non-$\mathbb{A}^1$-invariant setting, we do not expect a purity isomorphism, and in fact it does not hold in genera. Consider the example for $0 \hookrightarrow \mathbb{P}^1$, i.e.

$$\frac{\mathbb{P}^1}{\mathbb{P}^1 \setminus 0} \to \mathrm{Th}_0(N_{0/\mathbb{P}^1}) \simeq \frac{\mathbb{P}^1}{\infty}.$$

We see that this map is an equivalence if and only if $\mathbb{A}^1$ is contractible, hence certainly not in MS_S. However, we can still construct a map

$$\frac{X}{X \setminus Z} \to \mathrm{Th}_Z(N_{Z/X})$$

in general. Furthermore, this map induces the purity isomorphism upon $\mathbb{A}^1$-localisation.

Remark 10.4. It is already remarkable that the purity isomorphism is induced by a single map and not a zig-zag.

This is called the Gysin map of the closed embedding $Z \hookrightarrow X$.

10.1.1 Gysin maps

Once again, fix a closed embedding $Z \hookrightarrow X$ in Sm_S. Following Longke Tang, we will construct a Gysin map in MS_S of the form

$$\mathrm{gys}_{Z/X}\colon X_+ \to \mathrm{Th}_Z(N_{Z/X}).$$

First, consider the blowups as follows.

$$\begin{tikzcd} \mathbb{P}(N) \arrow[r] \arrow[d] & B_0 \arrow[d] \\ Z\times 0 \arrow[r, hook] & X \times 0 \end{tikzcd} \hookrightarrow \begin{tikzcd} \mathbb{P}(N\oplus\mathscr{O}) \arrow[r] \arrow[d] & B \arrow[d] \\ Z\times 0 \arrow[r, hook] & X \times \mathbb{P}^1. \end{tikzcd}$$

By smooth blow up excision, both squares become pushouts in MS_S. We can then take the cofibre of this map of pushout squares to obtain a new pushout square of the form

$$\begin{tikzcd} \mathrm{Th}_Z(N) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & B/B_0 \arrow[d] \\ 0 \arrow[r] & \frac{X\times\mathbb{P}^1}{X\times 0}. \end{tikzcd}$$

Now note that both blowups were along $Z\times 0$, whence we find a map $X\times 1 \to B/B_0$. Now $X\times 1$ and $X\times 0$ are homotopic inside $X\times\mathbb{P}^1$ by projective homotopy invariance, so the composite

$$X\times 1 \to B/B_0 \to \frac{X\times\mathbb{P}^1}{X\times 0}$$

admits a nullhomotopy, inducing a map to the fibre

$$\mathrm{gys}_{Z/X}\colon X\times 1 \to \mathrm{Th}_Z(N).$$

Remark 10.5. This is evidently functorial in the blowup, since the $\mathbb{P}^1$-homotopy chosen here is canonical. It then suffices to note that the blowup itself is functorial in excess intersection squares. More concretely, given an excess intersection square

$$\begin{tikzcd} Z' \arrow[r, hook] \arrow[d] & X' \arrow[d] \\ Z \arrow[r, hook] & X, \end{tikzcd}$$

we obtain a commuting diagram

$$\begin{tikzcd} X'_+ \arrow[r, "\mathrm{gys}_{Z'/X'}"] \arrow[d] & \mathrm{Th}_{Z'}(N_{Z'/X'}) \arrow[d] \\ X_+ \arrow[r, "\mathrm{gys}_{Z/X}"] & \mathrm{Th}_Z(N_{Z/X}). \end{tikzcd}$$

Furthermore, the construction of Gysin maps is Sm_S-linear, in the sense that for a fixed closed embedding $Z \hookrightarrow X$, it induces an Sm_S-linear functor

$$\mathrm{Sm}_X \to \mathrm{MS}_S^{\Delta^1}, Y \mapsto \mathrm{gys}_{Z/Y_Z}.$$

We can lift the Gysin maps to a natural transformation of functors Indeed, fixing a closed embedding as usual, let us label its edges by

$$\begin{tikzcd} Z \arrow[r, hook, "i"] \arrow[dr, "g"'] & X \arrow[d, "f"] \\ & S. \end{tikzcd}$$

Then note that the restriction map

$$\mathrm{Fun}_{\mathrm{MS}_S}(\mathrm{MS}_X, \mathrm{MS}_S) \to \mathrm{Fun}_{\mathrm{Sm}_S}(\mathrm{Sm}_X, \mathrm{MS}_S)$$

between MS_S-linear and Sm_S-linear functors is fully faithful. Indeed, MS_S was constructed from Sm_S by presheaves satisfying descent and invertibility of $\mathbb{P}^1$. The target contains the functors

$$Y \mapsto Y_+, \qquad\qquad Y \mapsto \mathrm{Th}_{Y_Z}(N_i),$$

which are (respectively) the images under restriction of the functors

$$f_\sharp \qquad\qquad g_\sharp \Sigma^{N_i} i^*.$$

By fully faithfulness, the map gys in the target therefore lifts to an MS_S-linear natural transformation

$$\mathrm{gys}\colon f_\sharp \to g_\sharp \Sigma^{N_i} i^*.$$

10.1.2 Key properties of Gysin maps

Let us now list some essential properties of Gysin maps which we will not prove (neither of them are obvious) but will use extensively later.

1. Let $z\colon S \hookrightarrow \mathbb{P}(\mathscr{E} \oplus \mathscr{O})$ be the zero section of the projectivisation of $\mathscr{E} \in \mathrm{Vect}(S)$. Then

   $$\mathrm{gys}_z\colon \mathbb{P}(\mathscr{E} \oplus \mathscr{O}) \to \mathrm{Th}_S(\mathscr{E})$$

   is precisely the quotient map killing $\mathbb{P}(\mathscr{E})$.

2. If $Z \hookrightarrow Y \hookrightarrow X$ is a composable pair of closed embeddings in Sm_S, then note that there is a fibre sequence in $\mathrm{Perf}(Z)$ of the form

   $$N_{Y/X}\mid_Z \to N_{Z/X} \to N_{Z/Y},$$

   which then splits in $\mathscr{K}(Z)$ and therefore also on Thom spectra, as these only depend on the K-theory class of a bundle. We then obtain a diagram

   $$\begin{tikzcd} X_+ \arrow[r, "\mathrm{gys}_{Y/X}"] \arrow[d, "\mathrm{gys}_{Z/X}"] & \mathrm{Th}_Y(N_{Y/X}) \arrow[d, "\mathrm{gys}_{Z/Y}"] \\ \mathrm{Th}_Z(N_{Z/X}) \arrow[r, "\sim"] & \mathrm{Th}_Z(N_{Z/Y} \oplus N_{Y/X}\mid_Z), \end{tikzcd}$$

   which commutes. Note that the right vertical arrow really comes from applying the Gysin natural transformation to $\mathrm{Th}_Y(N_{Y/X})$.

10.1.3 Atiyah duality

Let us now prove Theorem 10.1 using Gysin maps.

10.2 The comparison map

Let $x\colon X \to S$ be a smooth and separated morphism. In this section we construct a "norm map"

$$\alpha_f\colon f_\sharp \Sigma^{-\Omega_f} \to f_*.$$

However, note that just as in the construction of norm maps in equivariant homotopy theory, we can immediately reduce this to constructing norm maps for the diagonal. Consider the commutative diagram

$$\begin{tikzcd} X \arrow[r, hook, "\delta"] \arrow[dr, "\mathrm{id}_X"'] & X \times_S X \arrow[d, "\pi_2"] \\ & X, \end{tikzcd}$$

where δ is a closed embeding by the separated assumption. The associated Gysin transformation is then of the form

$$\mathrm{gys}_\delta \colon (\pi_2)_\sharp \to (\mathrm{id}_X)_\sharp \Sigma^{N_i} \delta^*.$$

Now note that the construction of a map α_f is–by adjunction–equivalence to constructin a map

$$\epsilon_f \colon f^* f_\sharp \Sigma^{-\Omega_f} \to \mathrm{id}.$$

By smooth base change for the diagram

$$\begin{tikzcd} X \times_S X \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & S, \end{tikzcd}$$

we can identify

$$\begin{aligned} f^* f_\sharp \Sigma^{-\Omega_f} &\simeq (\pi_2)_\sharp \pi_1^* \Sigma^{-\Omega_f}, \\ &\simeq (\pi_2)_\sharp \Sigma^{-\Omega_{\pi_2}} \pi_1^*, \\ &\xrightarrow{\mathrm{gys}_\delta} \Sigma^{N_\delta} \delta^* \Sigma^{-\Omega_{\pi_2}} \pi_1^*, \\ &\simeq \Sigma^{N_\delta - \delta^* \Omega_{\pi_2}} \delta^* \pi_1^*, \\ &\simeq \mathrm{id}. \end{aligned}$$

Indeed, the fibre sequence in $\mathrm{Perf}(X)$ of the form

$$N_\delta \to \delta^* \Omega_{\pi_2} \to \Omega_{\mathrm{id}_X}$$

and the fact that $\Omega_{\mathrm{id}_X} \simeq 0$ allows us to trivialise the twist at the end, and it is clear that $\pi_1 \delta = \mathrm{id}_X$.

Remark 10.6. Having constructed α_f in general, let us note some easy categorical observations that will help us prove when it is an equivalence.

- $\alpha_f \colon f_\sharp \Sigma^{-\Omega_f} \to f_*$ is an equivalence if and only if the adjoint map $\epsilon_f \colon f^* f_\sharp \Sigma^{-\Omega_f} \to \mathrm{id}$ is the counit map of an adjunction $f^* \dashv f_\sharp \Sigma^{-\Omega_f}$.
- Applying α_f to $\mathbb{1}_X$ gives a map $\mathrm{Th}_X(-\Omega_f) \to X_+^\vee$.
- Applying α_f to $\mathrm{Th}_X(\xi)$ for some $\xi \in \mathscr{K}(X)$ gives a map $\mathrm{Th}_X(-\Omega_f - \xi) \to \mathrm{Th}_X(\xi)^\vee$.

Let us now outline the proof strategy for showing that α_f is an equivalence whenever f is smooth and projective. The first two steps are entirely formal and categorical, while the third step is the heart of the proof.

1. Note that any smooth and proper map can be Zariski-locally factored through an embedding into a projective space

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[dr, "f"'] & \mathbb{P}^n \arrow[d, "p"] \\ & S. \end{tikzcd}$$

   In fact, α_f is an equivalence if α_p is an equivalence, so we only need to prove this for projective spaces.

Proof. If α_p is an equivalence, there is an adjunction $p^* \dashv p_\sharp \Sigma^{-\Omega_p}$, which also has a unit map

$$\eta_p \colon \mathrm{id} \to p_\sharp \Sigma^{-\Omega_p} p^*.$$

Now define

$$\eta_f \colon \mathrm{id} \xrightarrow{\eta_p} p_\sharp \Sigma^{-\Omega_p} p^* \xrightarrow{\mathrm{gys}_i} f_\sharp \Sigma^{-\Omega_f} f^*,$$

and check that η_f and ϵ_f satisfy the triangle identities, whence they form an adjunction and α_f is an equivalence. □

2. Suppose that $\Sigma^\infty_{\mathbb{P}^1} X_+ \in \mathrm{MS}_S$ is dualisable, and α_f is an equivalence when evaluated at $\mathbb{1}_X$, i.e. $\mathrm{Th}_X(-\Omega_f) \xrightarrow{\sim} X_+^\vee$. Then α_f is a natural equivalence. In particular, we know that all $\Sigma^\infty_{\mathbb{P}^1} \mathbb{P}^n$ are dualisable, since $\mathbb{P}^1$ is and the identification $\mathbb{P}^n/\mathbb{P}^{n-1} \simeq \mathbb{P}^1$ expresses $\mathbb{P}^n$ as an iterated extension of dualisables. By the point above it therefore suffices to check $\alpha_{\mathbb{P}^n}$ induces an equivalence $\mathrm{Th}_{\mathbb{P}^n}(-\Omega_{\mathbb{P}^n}) \xrightarrow{\sim} (\mathbb{P}^n_+)^\vee$.

   Proof. Assuming the above, first check that $\alpha_f f^*$ and $\alpha_f \Sigma^{\Omega_f} f^*$ are equivalences, this follows from the projection formula. Then use the triangle identities for $f^* \dashv f_*$ to see that α_f has left and right inverse natural transformations. □

3. By induction on n, prove that the map

$$\alpha_{\mathbb{P}^n} \colon \mathrm{Th}_{\mathbb{P}^n}(-\Omega_{\mathbb{P}^n} - \mathscr{O}(-1)^m) \to \mathrm{Th}_{\mathbb{P}^n}(\mathscr{O}(-1)^m)^\vee$$

   is an equivalence by induction. The next subsection is dedicated to proving this.

Remark 10.7. Note that $p_* \mathbb{1}_{\mathbb{P}^n} \simeq (\mathbb{P}^n_+)^\vee \simeq \underline{\mathrm{hom}}(\mathbb{P}^n_+, \mathbb{1}_S)$, so this last point is really a statement about the cohomology of projective space.

10.2.1 Duals of projective spaces

Let us first note that the (ξ-twisted) norm map

$$\alpha_f \colon \mathrm{Th}_X(-\Omega_f - \xi) \to \mathrm{Th}_X(\xi)^\vee$$

is bivariantly natural in X, ξ with respect to closed immersions in Sm_S.

- The covariant functoriality in $Z \hookrightarrow X$ means that there is a commutative diagram

$$\begin{tikzcd} \mathrm{Th}_Z(-\Omega_X|_Z - \xi|_Z) \arrow[r, "\alpha_Z"] \arrow[d, "\mathrm{inc}"] & \mathrm{Th}_Z(N_{Z/X} + \xi|_Z) \arrow[d, "\mathrm{gys}^\vee_{Z/X}"] \\ \mathrm{Th}_X(-\Omega_X - \xi) \arrow[r, "\alpha_X"] & \mathrm{Th}_X(\xi)^\vee, \end{tikzcd}$$

  where we identified the top right corner using the identification $\Omega_X|_Z \simeq \Omega_Z + N_{Z/X}$ in $\mathscr{K}(Z)$.

- The contravariant functoriality in $Z \hookrightarrow X$ means that there is a commutative diagram

$$\begin{tikzcd} \mathrm{Th}_X(-\Omega_X - \xi) \arrow[r, "\alpha_X"] \arrow[d, "\mathrm{gys}_{Z/X}"] & \mathrm{Th}_X(\xi)^\vee \arrow[d, "\mathrm{inc}^\vee"] \\ \mathrm{Th}_X(-\Omega_Z - \xi|_Z) \arrow[r, "\alpha_Z"] & \mathrm{Th}_X(\xi|_Z)^\vee. \end{tikzcd}$$

Both of these statements follows directly from the functoriality of Gysin maps. Let us now apply this bivariant functoriality to the pair of (disjoint) closed embeddings

$$\mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^n \hookleftarrow \mathbb{P}^0.$$

Note that $N_{\mathbb{P}^{n-1}/\mathbb{P}^n} \simeq \mathscr{O}(-1)$, so in particular

$$\Omega_{\mathbb{P}^n} |_{\mathbb{P}^{n-1}} \simeq \Omega_{\mathbb{P}^{n-1}} + \mathscr{O}(-1)$$

in $\mathscr{K}(\mathbb{P}^{n-1})$. The two squares expressing bivariant functoriality can then be glued together along $\alpha_{\mathbb{P}^n}$ into a commutative diagram

$$\begin{tikzcd}
\mathrm{Th}_{\mathbb{P}^{n-1}}(-\Omega_{\mathbb{P}^n} - \mathscr{O}(-1)) \arrow[r, "\mathrm{inc}"] \arrow[d, "\alpha_{\mathbb{P}^{n-1}}"] & \mathrm{Th}_{\mathbb{P}^n}(-\Omega_{\mathbb{P}^n}) \arrow[r, "\mathrm{gys}_{\mathbb{P}^0/\mathbb{P}^n}"] \arrow[d, "\alpha_{\mathbb{P}^n}"] & \mathrm{Th}_{\mathbb{P}^0}(\mathscr{O}^n - \mathscr{O}^n) \arrow[r, "\sim"] \arrow[d, "\alpha_{\mathbb{P}^0}"] & \mathbb{1}_S \arrow[d, "\sim"] \\
\mathrm{Th}_{\mathbb{P}^{n-1}}(\mathscr{O}(-1))^\vee \arrow[r, "\mathrm{gys}^\vee_{\mathbb{P}^{n-1}/\mathbb{P}^n}"] & (\mathbb{P}^n_+)^\vee \arrow[r, "\mathrm{inc}^\vee"] & (\mathbb{P}^0_+)^\vee \arrow[r, "\sim"] & \mathbb{1}_S.
\end{tikzcd}$$

We see that the rightmost vertical map is an equivalence for trivial reasons, so if we want to prove that $\alpha_{\mathbb{P}^n}$ is an equivalence by induction on n, it would suffice to prove that this diagram is actually a map of cofibre sequences by the five lemma.

Remark 10.8. Since $\mathbb{P}^{n-1}$ and $\mathbb{P}^0$ are disjoint, the horizontal composites factor over $\mathrm{Th}_\varnothing(\cdots)$ hence admit a canonical nullhomotopy. Furthermore, by the naturality of norm maps, the vertical arrows then assemble to a map of nullsequences. It therefore suffices to check that the horizontal parts of the diagram are cofibre sequences and this will then become a map of cofibre sequences.

Let us now axiomatise under which conditions this becomes a map of cofibre sequences.

Definition 10.9. Let $Y, X \hookrightarrow X$ be closed embeddings in Sm_S such that $Y \cap Z = \varnothing$, and fix a class $\xi \in \mathscr{K}(X)$. We say that (X, Y, Z, ξ) is a Gysin quadruple if the nullsequence

$$\mathrm{Th}_Y(\xi |_Y) \xrightarrow{\mathrm{inc}} \mathrm{Th}_X(\xi) \xrightarrow{\mathrm{gys}} \mathrm{Th}_X(N_{Z/X} + \xi_Z)$$

is a cofibre sequence in MS_S.

We can then rephrase the goal of this section as proving that the quadruples

$$(\mathbb{P}^n, \mathbb{P}^{n-1}, \mathbb{P}^0, -\Omega_{\mathbb{P}^n} - \mathscr{O}(-1)^m), \text{ and } (\mathbb{P}^n, \mathbb{P}^0, \mathbb{P}^{n-1}, \mathscr{O}(-1)^m) \tag{3}$$

are Gysin quadruples for all n, m.

Remark 10.10. We know the following are Gysin quadruples.

1. For any $\mathscr{E} \in \mathrm{Vect}(X)$, $(\mathbb{P}_X(\mathscr{E} \oplus \mathscr{O}), \mathbb{P}_X(\mathscr{E}), X, 0)$ is a Gysin quadruple by the normalisation property (1).
2. If η is a class in $\mathscr{K}(S)$, then (X, Y, Z, ξ) is a Gysin quadruple if and only if $(X, Y, Z, \xi + \eta)$ is a Gysin quadruple (identifying η with its pullback to X).
3. Let $B = \mathrm{Bl}_Y X$ be the blowup of X in Y with exceptional divisor E. Then smooth blowup excision shows that (X, Y, Z, ξ) is a Gysin quadruple if and only if (B, E, Z, ξ) is a Gysin quadruple.
4. Let D be a smooth divisor on X and $\mathscr{E} \in \mathrm{Vect}(X)$ a finite locally free sheaf[8]. Then $(X, D, Z, \mathscr{E})$ is a Gysin quadruple if and only if $(X, D, Z, \mathscr{E}(D))$ is a Gysin quadruple.

This last item is the key and is the only one that is not trivial to prove given what we have established above. It follows from a general cube argument with smooth normal crossing divisors analogously to the proof of multiplicativity of Thom spectra to establish a zig-zag of equivalences

$$\frac{\mathrm{Th}_X(\mathscr{E})}{\mathrm{Th}_D(\mathscr{E})} \simeq \frac{\mathrm{Th}_X(\mathscr{E}(D))}{\mathrm{Th}_D(\mathscr{E}(D))}.$$

We then prove the desired property (3) by applying items 3 to 4 sufficiently many times to item 1 above.

[8]It is not known whether $\mathscr{E}$ can be a general K-theory class

11 $\mathbb{A}^1$-colocalisation and logarithmic cohomology

We'll discuss a right adjoint to the inclusion of $\mathbb{A}^1$-invariant motivic spectra, and how it relates to logarithmic cohomology theories.

11.1 $\mathbb{A}^1$-colocalisation

Let us begin by remarking that the inclusion

$$\mathrm{MS}_S^{\mathbb{A}^1} \hookrightarrow \mathrm{MS}_S$$

admits both left and right adjoints.

Proof. By Nisnevich descent for MS, we can assume that S is qcqs. Now consider the inclusion

$$\mathrm{MS}_S^{(\mathbb{A}^1)} \hookrightarrow \mathrm{Sp}_{\mathbb{P}^1}^{\mathrm{lax}}(\mathscr{P}(\mathrm{Sm}_S; \mathrm{Sp})).$$

This is the inclusion of the full subcategory on spectra satisfying the following.

1. Nisnevich descent and elmentary blowup excision.
2. Strict $\mathbb{P}^1$-spectra, i.e. such that the map $E_i \to \Omega_{\mathbb{P}^1} E_{i+1}$ is an equivalence.
3. In the case of $\mathbb{A}^1$-invariant motivic spectra, that the map $X \times \mathbb{A}^1 \to X$ is an equivalence.

It now suffices to note that all of these conditions are closed under both limits and colimits.

1. Since we are working in the stable setting, and both Nisnevich descent and elementary blowup excision are conditions that involve sending certain squares to Cartesian squares.
2. Since the functor $\Omega_{\mathbb{P}^1}$ preserves both limits and colimits.
3. This is clear.

By presentability, we therefore have both adjoints

$$\begin{tikzcd} \mathrm{MS}_S^{\mathbb{A}^1} \arrow[r, hook] & \mathrm{MS}_S. \arrow[l, bend right, "L_{\mathbb{A}^1}"'] \arrow[l, bend left, "R_{\mathbb{A}^1}"] \end{tikzcd}$$

□

Remark 11.1. In fact, there is an explicit simplicial formula for $L_{\mathbb{A}^1}$:

$$L_{\mathbb{A}^1} E \simeq \left| E^{\mathbb{A}^\bullet} \right|.$$

Now note that $L_{\mathbb{A}^1}$ is a symmetric monoidal functor, whence its right adjoint (the inclusion) admits the structure of a lax symmetric monoidal functor, giving rise to a factorisation of this adjunction.

$$\begin{tikzcd} \mathrm{MS}_S^{\mathbb{A}^1} \arrow[r, hook] & \mathrm{Mod}(\mathrm{MS}_S; \mathbb{1}_{\mathbb{A}^1}). \arrow[l, bend right, "L_{\mathbb{A}^1}"'] \arrow[l, bend left, "(-)^\dagger"] \end{tikzcd}$$

Indeed, the forgetful functor $\mathrm{Mod}(\mathrm{MS}_S; \mathbb{1}_{\mathbb{A}^1}) \to \mathrm{MS}_S$ creates both limits and colimits. In this notation, $\mathbb{1}_{\mathbb{A}^1}$ is the image of the unit of $\mathrm{MS}_S^{\mathbb{A}^1}$ under the inclusion, or equivalently the localisation $L_{\mathbb{A}_1} \mathbb{1}_{\mathrm{MS}_S}$.

Remark 11.2. Since $L_{\mathbb{A}^1}$ was symmetric monoidal, we concluded that the inclusion was lax symmetric monoidal, but we can not obtain any information about the further right adjoint. However, these functors turn out to be linear over $\mathrm{MS}_S^{\mathrm{du}}$.

Definition 11.3. Let $\mathscr{C} \in \mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}})$. Let $\mathscr{C}^{\mathrm{du}} \subset \mathscr{C}$ denote the full subcategory on dualisable objects, and let $\mathscr{C}^{\mathrm{lisse}}$ denote its closure under colimits in $\mathscr{C}$.

Remark 11.4. If $\mathscr{C}^{\mathrm{du}} \subset \mathscr{C}^{\omega}$, i.e. dualisable objects are compact, then

$$\mathscr{C}^{\mathrm{lisse}} \simeq \mathrm{Ind}(\mathscr{C}^{\mathrm{du}}) \hookrightarrow \mathscr{C}.$$

This is indeed the case for many categories of interest to us, e.g. MS_S over a qcqs base[9].

Lemma 11.5. *If S is qcqs, then every lisse $\mathbb{1}_{\mathbb{A}^1}$-module in MS_S is $\mathbb{A}^1$-invariant.*

Proof. Let E be a lisse $\mathbb{1}_{\mathbb{A}^1}$ module, then we want to show that for all $X \in \mathrm{Sm}_S^{\mathrm{fp}}$, the map

$$E(X) \to E(\mathbb{A}^1 \times X)$$

is an equivalence. By the remark above, since dualisables are compact in MS_S, E is a filtered colimit of dualisables, we can assume that E is dualisable. Then

$$\begin{aligned} E(X) &\simeq \mathrm{Map}_{\mathbb{1}_{\mathbb{A}^1}}(\mathbb{1}_{\mathbb{A}^1} \otimes \Sigma^\infty_{\mathbb{P}^1} X_+, E), \\ &\simeq \mathrm{Map}_{\mathbb{1}_{\mathbb{A}^1}}(E^\vee \otimes \Sigma^\infty_{\mathbb{P}^1} X_+, \mathbb{1}_{\mathbb{A}^1}), \\ &\simeq \mathrm{Map}_{\mathrm{MS}_S^{\mathbb{A}^1}}(L_{\mathbb{A}^1}(E^\vee \otimes \Sigma^\infty_{\mathbb{P}^1} X_+), \mathbb{1}_{\mathbb{A}^1}). \end{aligned}$$

The last term is clearly $\mathbb{A}^1$-invariant. □

Corollary 11.6.

- *If $E \in \mathrm{MS}_S^{\mathrm{lisse}}$ is lisse, then $\mathbb{A}^1$-localisation is smashing, i.e.*

$$L_{\mathbb{A}^1} E \simeq \mathbb{1}_{\mathbb{A}^1} \otimes E.$$

- *Let E be a module over $\mathbb{1}_{\mathbb{A}^1}$, then the counit map*

$$E^\dagger \to E$$

induces an equivalence on lisee objects, i.e. for all $X \in \mathrm{MS}_S^{\mathrm{lisse}}$, we have

$$E^\dagger(X) \xrightarrow{\sim} E(X).$$

The idea is then to provide a wealth of $\mathbb{1}_{\mathbb{A}^1}$-modules and lisse objects on which to apply this equivalence.

Remark 11.7. By Atiyah duality, for any smooth projective S-scheme X and K-theory class $\xi \in \mathscr{K}(X)$, $\mathrm{Th}_X(\xi)$ is dualisable in MS_S.

In particular, we see that if E is a $\mathbb{1}_{\mathbb{A}^1}$-module, its value on $\mathrm{Th}_X(\xi)$ as above does not depend on whether we consider E or its $\mathbb{A}^1$-colocalisation $E^\dagger$. Let us now supply examples of such $\mathbb{1}_{\mathbb{A}^1}$-modules.

Example 11.8. Recall that we outlined a procedure that associated to any localising invariant

$$E \in \mathrm{Fun}^{\mathrm{loc}}(\mathcal{C}\mathrm{at}_{\mathrm{ex}}, \mathrm{Sp})$$

a motivic spectrum $E_S \in \mathrm{MS}_S$, which in fact lands in $\mathrm{Mod}(\mathrm{MS}_S; \mathrm{KGL}_S)$ since K-theory is the universal localising invariant.

[9] For this reason, we carry this assumption throughout the following corollaries

1. If S is regular and Noetherian, then K-theory is $\mathbb{A}^1$-invariant, so $\mathrm{KGL}_S \in \mathrm{MS}_S^{\mathbb{A}^1}$ and as a consequence, for any localising invariant E, E_S is a $\mathbb{1}_{\mathbb{A}^1}$-module.

2. Over a general base scheme, if E is a truncating localising invariant, then Land–Tamme show that the restriction of E to schemes is a module over the cdh sheafification of K-theory, which itself is just Weibel's homotopy K-theory. Therefore, E_S is a $L_{\mathbb{A}^1}\mathrm{KGL}$-module, hence also a $\mathbb{1}_{\mathbb{A}^1}$-module. An example of a truncating invariant is the fibre $\mathrm{K}^{\mathrm{inv}}$ of the cyclotomic trace by the Dundas–Goodwillie–McCarthy theorem.

3. Every rational oriented motivic spectrum over a regular base is a $\mathbb{1}_{\mathbb{A}^1}$-module.

4. If S is a Dedekind scheme, we used Voevodsky's slice filtration on K-theory as well as motivic filtrations on localising invariants to construct motivic spectra

$$\begin{tikzcd} & & & \mathrm{thh}_p \\ \mathrm{kgl} \arrow[r] & \mathrm{tc}_p \arrow[r] & \mathrm{tc}_p^- \arrow[ur] \arrow[dr] & \\ & & & \mathrm{tp}_p \end{tikzcd}$$

for any prime p. Since we are working over a Dedekind scheme, kgl will be $\mathbb{A}^1$-invariant, making all of these into $\mathbb{1}_{\mathbb{A}^1}$-modules. Further, taking associated graded (i.e. killing β in these motivic spectra) gives us more $\mathbb{1}_{\mathbb{A}^1}$-modules[10]

$$\begin{tikzcd} & & & \mathrm{gr}_*^{\mathrm{N}}\mathrm{H}\mathbb{Z}_p^{\Delta} \\ \mathrm{H}\mathbb{Z} \arrow[r] & \mathrm{H}\mathbb{Z}_p^{\widehat{\mathrm{syn}}} \arrow[r] & \mathrm{Fil}_\star^{\mathrm{N}}\mathrm{H}\mathbb{Z}_p^{\Delta} \arrow[ur] \arrow[dr] & \\ & & & \mathrm{H}\mathbb{Z}_p^{\Delta}. \end{tikzcd}$$

5. Over a field, we don't know of a motivic spectrum that is not a $\mathbb{1}_{\mathbb{A}^1}$-module.

11.2 Logarithmic cohomology

Instead of defining logairhtmic cohomology as an invariant of log schemes, let us restrict ourselves to the case of smooth normal crossing divisors. Let $X \in \mathrm{Sm}_S$ be a smooth S-scheme, and ∂X a relative SNCD on X. Denote its components by $\partial_1 X, \dots, \partial_n X$, and note that per definition, the intersection

$$\forall I \subseteq \underline{n}, \partial_I X := \bigcap_{i\in I} \partial_i X$$

is still a smooth S-scheme. We use this to define a functor

$$\partial_- X \colon \square^{n,\mathrm{op}} \to \mathrm{Sm}_S, I \mapsto \partial_I X,$$

where $\square^n$ denotes the n-cube, i.e. the poset of subsets of $\underline{n}$. We further let N_I denote the conormal sheaf of $\partial_I X \hookrightarrow X$.

[10] A priori the associated gradeds on tc_p^- and tp_p only recover Nygaard completed prismatic cohomology (and its Nygaard filtration), but since we are working over a Dedekind base there is not difference.

Definition 11.9. Define a functor
$$\square^n \to \mathrm{MS}_S, I \mapsto \mathrm{Th}_{\partial_I X}(N_I).$$
Its functoriality is given by the Gysin maps
$$(I \subseteq J) \mapsto (\mathrm{Th}_{\partial_I X}(N_I) \xrightarrow{\mathrm{gys}} \mathrm{Th}_{\partial_J X}(N_J)).$$

Remark 11.10. For the definition above to make sense as stated, we require more coherent compatibility between Gysin maps than we proved. This can be done, albeit laboriously, but it can also be circumvented by assumping that X is projective. In that case, we can use Atiyah duality to give an equivalence $\mathrm{Th}_{\partial I X}(-\Omega X)^\vee \simeq \partial_I X$, such that the Gysin maps above can be identified with the duals of the inclusion maps, which are obviously maximally functorial.

This allows us to define logarithmic cohomology theories associated to motivic spectra.

Definition 11.11. Let $E \in \mathrm{MS}_S$ be a motivic spectrum and $(X, \partial X)$ a SNCD in Sm_S. Define
$$E_{\log}(X, \partial X) := \mathrm{tcof}(\square^{n,\mathrm{op}} \to \mathrm{MS}_S^{\mathrm{op}} \xrightarrow{E} \mathrm{Sp})$$
in terms of the construction above.

Remark 11.12. Since the total cofibre of a cube is defined as the cofibre of the map from the pushout of the punctured cube with its final vertex removed to the value at the final vertex, we obtain a fibre sequence
$$\varinjlim_{\varnothing \neq I \in \square^n} E(\mathrm{Th}_{\partial_I X}(N_I)) \to E(X) \to E_{\log}(X, \partial X),$$
since the initial vertex (hence the final vertex in the opposite cube) is precislely $I = \varnothing$ with $\partial_\varnothing X = X$.

This also enjoys the invariance property of $\mathbb{A}^1$-colocalisation on smooth projective S-schemes. Indeed, if X is a smooth and projective, then so is every $\partial_I X$. If E is a $\mathbb{1}_{\mathbb{A}^1}$-module, we can apply (the twisted variant of) Corollary 11.6 to the cofibre sequence in Remark 11.12 to obtain an equivalence
$$E_{\log}(X, \partial X) \simeq E^\dagger_{\log}(X, \partial X).$$

Proposition 11.13. *Let $(X, \partial X)$ be a smooth projective SNCD compactification of some smooth projective S-scheme U. Then there is an equivalence*
$$E_{\log}(X, \partial X) \simeq E^\dagger(U),$$
as well as twisted variants for any K-theory class on U. In particular, the left hand side is independent of the choice of SNCD compactification of U.

Proof. By the discussion above, and the restriction to smooth projective schemes, we see that $E_{\log}(X, \partial X)$ is equivalent to $E^\dagger_{\log}(X, \partial X)$. Therefore, both sides of the desired equivalence are computed $\mathbb{A}^1$-invariant motivic spectra. It then suffices to show that the map
$$U_+ \to \mathrm{tfib}(\square^n \xrightarrow{\mathrm{Th}_{\partial_- X}(N_-)} \mathrm{MS}_S)$$
is an $\mathbb{A}^1$-equivalence. Since $\mathbb{A}^1$-equivalences are precisely those detected by $L_{\mathbb{A}^1}$, it suffices to check that this is an equivalence on $\mathbb{A}^1$-invariant motives, i.e. in $\mathrm{MS}_S^{\mathbb{A}^1} \simeq \mathrm{SH}(S)$. There, it is a consequence of Morel–Voevodsky's purity isomorphism. We sketch the proof in the case $n = 1$, so that the total fibre is just a fibre, and denote $\partial X = \partial_1 X$ by Z. The purity isomorphism then gives us a fibre sequence
$$U_+ \to X_+ \xrightarrow{\mathrm{gys}} \mathrm{Th}_X(N_{Z/X})$$
in $\mathrm{SH}(S)$ and we are done. □

Let us now list some example applications of the Proposition above.

Theorem 11.14. *Let k be a perfect field of characteristic p, and consider the motivic spectrum*

$$\mathrm{H}W(k)^{\mathrm{crys}} = \mathrm{H}(W(k),(p))^{\Delta} \in \mathrm{MS}_k$$

representing crystalline cohomology, or equivalently, prismatic cohomology relative to the crystalline prism $(W(k),(p))$. Note that this is a $\mathbb{1}_{\mathbb{A}^1}$-module, since it is a module over the absolute prismatic cohomology spectrum $\mathrm{H}\mathbb{Z}^{\Delta}$, which itself is a module over tp_p, and we are in the situation of item 4 of Example 11.8.

1. *The $\mathbb{A}^1$-colocalisation $\mathrm{H}W(k)^{\mathrm{crys},\dagger}$ is an integral refinement of Berthelot's rigid cohomology, i.e. there is an equivalence*
$$\mathrm{H}W(k)^{\mathrm{crys},\dagger}[1/p] \simeq \mathrm{H}W(k)^{\mathrm{rig}}.$$
2. *If $U \in \mathrm{Sm}_k$ admits an SNCD compactification $(X,\partial X)$, then there is an equivalence*
$$\mathrm{H}W(k)^{\mathrm{crys},\dagger}(U) \simeq \mathrm{R}\Gamma_{\mathrm{crys}}((X,\partial X)/W(k))$$
with Kato's logarithmic crystalline cohomology, in particular the right hand side is independent of the choice of compactification of U.

Remark 11.15.

- The fact that Kato's logarithmic crystalling cohomology does not depend on the choice of compactification was known assuming resolution of singularities (R.o.S.) over k. Indeed, in that case the category of compactifications is filtered.
- An integral refinement of rigid cohomology was first constructed by Ertl–Shiho–Sprang in 2021 assuming R.o.S. over k.
- Merici then constructed said integral refinement without assuming R.o.S. in 2024, but proved the independence of the choice of compactification only assuming R.o.S. over k.

Proof.

1. First, note that per construction $\mathrm{H}W(k)^{\mathrm{rig}}$ is $\mathbb{A}^1$-invariant. Therefore, the comparison map to p-inverted crystalline cohomology factors through the $\mathbb{A}^1$-colocalisation of the latter, producing a map[11]
$$\mathrm{H}W(k)^{\mathrm{rig}} \to (\mathrm{H}W(k)^{\mathrm{crys}}[1/p])^{\dagger}[1/p].$$
The comparison map
$$\mathrm{H}W(k)^{\mathrm{crys},\dagger}[1/p] \to (\mathrm{H}W(k)^{\mathrm{crys}}[1/p])^{\dagger}[1/p]$$
in $\mathrm{MS}_k^{\mathbb{A}^1}[1/p]$ is an equivalence on all smooth projective schemes (and their twists by Thom classes). We then apply a result by Levine–Yang–Zhao and Riou which states that $\mathrm{SH}(k)[1/p]$ is generated under colimits by (Thom twists of) smooth projective k-schemes. This allows us to conclude that both maps above are equivalences.
2. Note that $\mathrm{H}W(k)^{\mathrm{crys}}$ is oriented, so that we can use Proposition 11.13 and trivialise the relevant Thom twists to rewrite the left hand side as
$$\mathrm{H}W(k)^{\mathrm{crys},\dagger} \simeq \mathrm{tcof}_I\, \mathrm{H}W(k)^{\mathrm{crys}}(\Sigma^{\infty-d}_{\mathbb{P}^1}\partial_I X)^{\vee}).$$
Now note that crystalline cohomology satisfies the Künneth formula, which implies that
$$\mathrm{H}W(k)^{\mathrm{crys}}\colon \mathrm{MS}_k^{\mathrm{op}} \to \mathscr{D}(W(k))$$

[11] It maps to the right hand side before inverting p on the outside, but we know that rigid cohomology lands in $\mathbb{Q}_p$-vector spaces so there is no harm in further mapping to the p-inversion.

is symmetric monoidal, in fact the unique symmetric monoidal extension of $\mathrm{R}\Gamma_{\mathrm{crys}}$ from Sm_k. In particular, it preserves duals so that we can rewrite the formula above as

$$\mathrm{H}W(k)^{\mathrm{crys},\dagger}(U)^\vee \simeq \operatorname{tfib}_I \mathrm{R}\Gamma_{\mathrm{crys}}(\partial_I X/W(k))[2d].$$

We can now apply Poincaré duality in crystalline cohomology to identify the dual of the right hand side with compactly supported logarithmic crystalline cohomology

$$\mathrm{R}\Gamma_{\mathrm{crys},c}((X,\partial X)/W(k))[2d].$$

Nakkajima and Shiho provide an equivalence

$$\mathrm{R}\Gamma_{\mathrm{crys},c}((X,\partial X)/W(k)) \simeq \operatorname{tfib}_I \mathrm{R}\Gamma_{\mathrm{crys}}(\partial_I X/W(k)).$$

□

Remark 11.16. Levine–Yang–Zhao–Riou's result about $\mathrm{SH}(k)[1/p]$ uses Gabber alterations, a weakened form of R.o.S., which is known to be true over all perfect fields of characteristic p.

Remark 11.17. The recipe in the second step is sufficiently formal that it could be applied to a wider class of cohomology theories for which we have a Künneth formula, Poincaré duality, and a description of the compactly supported logarithmic theory, in particular if the latter two were established for de Rham cohomology, this proof would work as well.

12 Motivic Landweber exactness

In this talk, extend the formalism of Landweber exactness in SH due to Naumann–Spitzweck–Østvær to the non-$\mathbb{A}^1$-invariant setting. Given a graded formal group law F over a graded ring R, classified by a map $L \to R$ from the Lazard ring, the goal of this discussion is to construct a motivic spectrum $E_F \in \mathrm{MS}_S$ such that

$$(E_F)_*(-) \cong \mathrm{MGL}_*(-) \otimes_L R \colon \mathrm{MS}_S \to \mathrm{Ab}^{\mathrm{Gr}}.$$

12.1 Recollection on formal groups and MGL

1. Recall that we had an explicit presentation for MGL as MGr, i.e.

$$\mathrm{MGL} \simeq \varinjlim_n \Sigma^{\infty-n}_{\mathbb{P}^1} \mathrm{Th}_{\mathrm{Gr}_n}(Q_n).$$

Each (Thom spectrum over a) Grassmannian can further be expressed as a filtered colimit of (Thom spectra over) smooth projective schemes, namely by filtering it by the finite Grassmannians $\mathrm{Th}_{\mathrm{Gr}_{n,k}}(Q_{n,k})$. By Atiyah duality, the latter are all dualisable motivic spectra, so we conclude that

$$\mathrm{MGL} \in \mathrm{MS}_S^{\mathrm{lisse}}.$$

Remark 12.1. It is easier to show that KGL is lisse, since we have already proven the motivic Snaith theorem. Indeed, this states

$$\mathrm{KGL} \simeq \Sigma^\infty_{\mathbb{P}^1} \mathrm{Pic}_+[\beta^{-1}] \simeq \Sigma^\infty_{\mathbb{P}^1} \mathbb{P}^\infty[\beta^\pm],$$

and the right hand side is clearly lisse.

2. Further, recall that $\mathscr{M}_{\mathrm{fg}}$, the moduli of formal groups, is the base of a $\mathbb{G}_m$-torsor $\mathscr{M}_{\mathrm{fg}}^s$ parametrising coordinatisable formal groups such that both squares in the diagram of $\mathbb{G}_m$-torsors

$$\begin{tikzcd} \mathrm{Spec}(L) \arrow[r] \arrow[d] & \mathscr{M}_{\mathrm{fg}}^s \arrow[r] \arrow[d] & \mathrm{Spec}(\mathbb{Z}) \arrow[d] \\ \mathrm{Spec}(L)/\mathbb{G}_m \arrow[r] & \mathscr{M}_{\mathrm{fg}} \arrow[r] & B\mathbb{G}_m \end{tikzcd}$$

are Cartesian. The map $\mathscr{M}_{\mathrm{fg}} \to B\mathbb{G}_m$ classifies the dualising line ω of the universal formal group. Furthermore, the maps

$$\mathrm{Spec}(L) \to \mathscr{M}_{\mathrm{fg}}^s, \qquad \mathrm{Spec}(L)/\mathbb{G}_m \to \mathscr{M}_{\mathrm{fg}}$$

are faithfully flat maps, with the firswt one being faithfully flat and affine. In fact, taking the Čech nerve of this map gives us a simplicial presentation

$$\mathscr{M}_{\mathrm{fg}}^s \simeq \operatorname*{colim}_{\Delta^{\mathrm{op}}} \ \mathrm{Spec}(L) \leftleftarrows \mathrm{Spec}(LB) \;\substack{\longleftarrow\\\longleftarrow\\\longleftarrow}\; \cdots$$

in the category of prestacks Fun(CRing, Ani), even before sheafification.

3. the cooperations in MGL were computed as

$$(\mathrm{MGL} \otimes \mathrm{MGL})_* \cong \mathrm{MGL}_*[b_0, b_1, b_2, \ldots]$$

with the convention $b_0 = 1$. The right hand side is clearly flat over MGL_*, so this defined a Hopf algebroid where the left and right units are related by

$$c_R = \sum_{i=0}^{\infty} b_i c_L^{i+1} \in (\mathrm{MGL} \otimes \mathrm{MGL})^1(\mathrm{Pic}).$$

We can therefore define a map

$$(L, LB) \to (\mathrm{MGL}_*, \mathrm{MGL}_*\mathrm{MGL})$$

purely formally, by sending $b_i \in LB$ to b_i in the right hand side. If the base S is qcqs, then this is in fact a coCartesian morphism of Hopf algebroids, i.e. a functor of cosimplicial objects in graded rings, where all naturality squares are coCartesian.

Using this last point, we can naturally view $(\mathrm{MGL}_*, \mathrm{MGL}_*\mathrm{MGL})$-comodules as (L, LB)-comodules.

Corollary 12.2. *There is a factorisation of the form*

$$\begin{tikzcd} \mathrm{MS}_S \arrow[r, "\mathrm{MGL}(-)"] \arrow[dr, dashed, "\mathrm{MGL}_{\mathrm{fg}}(-)"'] & \mathrm{Ab}^{\mathrm{Gr}} \\ & \mathrm{Mod}^{\mathrm{Gr}}_{(L,LB)}. \arrow[u] \end{tikzcd}$$

The target of $\mathrm{MGL}_{\mathrm{fg}}$ *is the category of graded* (L, LB)*-comodules, which can be further identified as*

$$\mathrm{Mod}^{\mathrm{Gr}}_{(L,LB)} \simeq \mathrm{QCoh}(\mathscr{M}_{\mathrm{fg}})^{\heartsuit}.$$

12.1.1 The classical Landweber criterion

Proposition 12.3. *For a fixed prime* p*, we can inductively define canonical sections*

$$v_n \in \Gamma(\mathscr{M}_{\mathrm{fg}}^{\geq n}; \omega^{\otimes(p^n-1)})$$

with $\mathscr{M}_{\mathrm{fg}}^{\geq n}$ *the vanishing locus of* $v_0, \ldots, v_{n-1}$ *and the convention that* $v_0 = p$*.*

Example 12.4. Per construction, $\mathscr{M}_{\mathrm{fg}}^{\geq 0} \simeq \mathscr{M}_{\mathrm{fg}}$, while $\mathscr{M}_{\mathrm{fg}}^{\geq 1} \simeq \mathscr{M}_{\mathrm{fg}} \times \mathrm{Spec}(\mathbb{F}_p)$ is the vanishing locus of $v_0 = p$.

Remark 12.5. It is well known that the classes v_i in L are not canonical, but they are canonical as sections over the closed substacks as above, it is only their extension to all of $\mathscr{M}_{\mathrm{fg}}$ that is not unique. In fact, they can be described more explicitly. Given a formal group law F over a ring R, we obtain a pullback

$$\begin{tikzcd} \mathrm{Spec}(R/I_n) \arrow[rr] \arrow[d, hook] & & \mathscr{M}_{\mathrm{fg}}^{\geq n} \arrow[d, hook] \\ \mathrm{Spec}(R) \arrow[r, "F"] & \mathrm{Spec}(L) \arrow[r] & \mathscr{M}_{\mathrm{fg}}, \end{tikzcd}$$

since F factors through $\mathscr{M}_{\mathrm{fg}}^{s}$ (per construction) hence trivialises ω so that we can view the v_n's as global sections of the structure sheaf. In that case, $v_n \in R/I_n$ is canonically defined as the coefficient of x^{p^n} in the p-series $[p]_F(x)$.

We are now ready to state Landweber's criterion.

Theorem 12.6 (Landweber)**.** *Let* $\mathscr{F} \in \mathrm{QCoh}(\mathscr{M}_{\mathrm{fg}})^{\heartsuit}$ *be a quasicoherent sheaf on* $\mathscr{M}_{\mathrm{fg}}$*. Then* $\mathscr{F}$ *is flat if and only if for all primes* p*, the sequence* $(v_0, v_1, \ldots)$ *is regular on* $\mathscr{F}$*, i.e. for all* $n \geq 0$*, the map*

$$v_n \colon \mathscr{F}^{\geq n} \to \mathscr{F}^{\geq n} \otimes \omega^{\otimes(p^n-1)}$$

is injective.

Remark 12.7. One implication is clear, indeed if $\mathscr{F}$ is flat then so is its restriction to $\mathscr{M}_{\mathrm{fg}}^{\geq n}$, and tensoring with it therefore preserves injectivity of the map

$$v_n \colon \mathscr{O}^{\geq n} \to \omega^{\otimes(p^n-1)}.$$

Example 12.8.

- The map $\widehat{\mathbb{G}}_m \colon \operatorname{Spec}(\mathbb{Z}) \to \mathscr{M}_{\mathrm{fg}}$ classifying the multiplicative formal group is flat. Indeed, a choice of coordinates on $\widehat{\mathbb{G}}_m$ gives us the formal group law

$$f_m(x,y) = x + y \pm xy$$

with the sign depending on the choice of coordinate. However, it is clear that $v_0 = p$ so that $v_1 = 1$ is well defined modulo p[12].

- The map $\widehat{\mathbb{G}}_a \colon \operatorname{Spec}(\mathbb{Z}) \to \mathscr{M}_{\mathrm{fg}}$ classifying the addtive formal group is not Landweber exact. Indeed, $v_0 = p$, while $v_1 = 0$, and this is not a regular element in $\mathbb{F}_p$.
- The map $\mathscr{M}_{\mathrm{ell}} \to \mathscr{M}_{\mathrm{fg}}$ sending an elliptic curve to its formal completion at the identity is flat.[13]

12.2 The motivic Landweber exact functor theorem

Definition 12.9. Recall that we constructed a refinement $\mathrm{MGL}_{\mathrm{fg}}(-)$ of $\mathrm{MGL}_*(-)$ that landed in $\mathrm{QCoh}(\mathscr{M}_{\mathrm{fg}})^{\heartsuit}$. Let Φ_* denote the functor[14]

$$\Phi_* \colon \mathrm{QCoh}(\mathscr{M}_{\mathrm{fg}})^{\heartsuit} \to \mathrm{Fun}(\mathrm{MS}_S, \mathrm{Ab}^{\mathrm{Gr}}), \mathscr{F} \mapsto \Gamma(\mathscr{M}_{\mathrm{fg}}; \mathrm{MGL}_{\mathrm{fg}}(-) \otimes \mathscr{F} \otimes \omega^{\otimes *}).$$

Remark 12.10. From the definition of Φ_* as a composite, we can pinpoint to what extent an element in its image is a homological functor.

- First, $\mathrm{MGL}_{\mathrm{fg}}(-) \colon \mathrm{MS}_S \to \mathrm{QCoh}(\mathscr{M}_{\mathrm{fg}})^{\heartsuit}$ is homological per construction.
- If $\mathscr{F} \in \mathrm{QCoh}(\mathscr{M}_{\mathrm{fg}})^{\heartsuit}$ is flat, i.e. satisfies the Landweber criterion, then $\mathrm{MGL}_{\mathrm{fg}}(-) \otimes \mathscr{F}$ is still a homological functor.
- Tensoring with powers of ω to obtain the graded nature of the target is exact as ω is invertible.
- The global sections fucntor $\Gamma(\mathscr{M}_{\mathrm{fg}}; -)$ is not exact, in fact we should only expect global sections to be exact on quasicoherent sheaves that are pushed forward from affines.

Let us illustrate this last remark with an example.

Example 12.11. Consider a graded formal group law F over a graded ring R, classifying a map

$$\pi \colon \operatorname{Spec}(R)/\mathbb{G}_m \xrightarrow{F} \operatorname{Spec}(L)/\mathbb{G}_m \to \mathscr{M}_{\mathrm{fg}}.$$

Then for any graded R-module M we have $\pi_* M$ and we can explicitly identify

$$\Phi_* \pi_* M = \mathrm{MGL}_{\mathrm{fg}}(-) \otimes_L M,$$

which is homological if $\pi_* M$ was flat.

The example above gives a method to construct homology theories from formal group laws, but this is unnatural: the full LEFT depends only on a formal group without a choice of coordinate. Indeed, we should expect further functoriality and monoidality that is only visible if we obtain a coordinate-free construction.

[12] In fact, if we work away from the prime p the coordinate will always be such that the formal group law is given by $x + y + xy$, only at the prime two is there ramification giving us the different possible signs, but obviously these agree modulo two!

[13] This is key in constructing elliptic cohomology theories using the LEFT, but it is still very far from allowing us to construct topological modular forms.

[14] The asterisk here stands for grading.

Definition 12.12. Let $\mathrm{Mod}_{\mathrm{fg}}$ be the $(2,1)$-category of pairs

$$(\pi\colon \mathscr{X} \to \mathscr{M}_{\mathrm{fg}}, \mathscr{F})$$

for $\mathscr{X}\colon \mathrm{CRing} \to \mathrm{Ani}$ a prestack over $\mathscr{M}_{\mathrm{fg}}$ and $\mathscr{F} \in \mathrm{QCoh}(\mathscr{X})^{\heartsuit}$ a quasicoherent sheaf[15] on $\mathscr{X}$. A morphism from $(\mathscr{X},\mathscr{F})$ to $(\mathscr{Y},\mathscr{G})$ is given by a morphism $f\colon \mathscr{X} \to \mathscr{Y}$ over $\mathscr{M}_{\mathrm{fg}}$ and a morphism $f^*\mathscr{F} \to \mathscr{G}$.

Further, consider the full subcategory $\mathrm{Mod}_{\mathrm{fg}}^{\flat} \subset \mathrm{Mod}_{\mathrm{fg}}$ on pairs $(\mathscr{X},\mathscr{F})$ such that

- the pushforward $\pi_*\mathscr{F}$ is flat over $\mathscr{M}_{\mathrm{fg}}$, and
- $\mathscr{X}^s := \mathscr{X} \times_{\mathscr{M}_{\mathrm{fg}}} \mathscr{M}_{\mathrm{fg}}^s$ is affine, so that $\mathscr{X}$ is of the form $\mathrm{Spec}(R)/\mathbb{G}_m$.

The latter is clearly seen to be a coordinate-free decompletion of the category of flat (graded) formal group laws over affine schemes.

Remark 12.13. $\mathrm{Mod}_{\mathrm{fg}}^{\flat}$ is actually a $(1,1)$-category.

The key observation is now the following.

Lemma 12.14. *For $(\mathscr{X},\mathscr{F}) \in \mathrm{Mod}_{\mathrm{fg}}^{\flat}$ fitting into the (Cartesian) commutative diagram*

$$\begin{tikzcd} \mathscr{X}^s \arrow[r, "\pi^s"] \arrow[d, "q"] & \mathscr{M}_{\mathrm{fg}}^s \arrow[d, "p"] \\ \mathscr{X} \arrow[r, "\pi"] & \mathscr{M}_{\mathrm{fg}}, \end{tikzcd}$$

the functor

$$\Gamma(\mathscr{M}_{\mathrm{fg}}^s; p^*(- \otimes \pi_*\mathscr{F}))\colon \mathrm{QCoh}(\mathscr{M}_{\mathrm{fg}})^{\heartsuit} \to \mathrm{Ab}^{\mathrm{Gr}}$$

is exact.

Proof. By applying the projection formula and base change, we can rewrite the functor in question as

$$\begin{aligned} \Gamma(\mathscr{M}_{\mathrm{fg}}^s; p^*(- \otimes \pi_*\mathscr{F})) &\simeq \Gamma(\mathscr{M}_{\mathrm{fg}}^s; p^*\pi_*(\pi^*(-) \otimes \mathscr{F})), \\ &\simeq \Gamma(\mathscr{M}_{\mathrm{fg}}^s; \pi^s_* q^*(\pi^*(-) \otimes \mathscr{F})), \\ &\simeq \Gamma(\mathscr{X}^s; q^*(\pi^*(-) \otimes \mathscr{F})). \end{aligned}$$

Now observe the following.

- By assumption, $\mathscr{X}^s$ is affine, so $\Gamma(\mathscr{X}^s; -)$ is exact.
- q is pulled back from p hence also a $\mathbb{G}_m$-torsor so that q^* is exact.
- By assumption, $\pi_*\mathscr{F}$ is flat so that $- \otimes \pi_*\mathscr{F} \simeq \pi_*(\pi^*(-) \otimes \mathscr{F})$ is exact. Since $\mathscr{M}_{\mathrm{fg}}^s$ has affine diagonal and π is an affine map, π_* detects exactness so that $\pi^*(-) \otimes \mathscr{F}$ is exact.

□

Corollary 12.15. *If $(\mathscr{X},\mathscr{F}) \in \mathrm{Mod}_{\mathrm{fg}}^{\flat}$, then*

$$\Phi_*(\mathscr{X},\mathscr{F}) = \Phi_*\pi_*\mathscr{F} = \Gamma(\mathscr{M}_{\mathrm{fg}}; \mathrm{MGL}_{\mathrm{fg}}(-) \otimes \pi_*\mathscr{F} \otimes \omega^{\otimes^*})\colon \mathrm{MS}_S \to \mathrm{Ab}^{\mathrm{Gr}}$$

is a homological functor.

[15] It is not entirely clear what $\mathrm{QCoh}^{\heartsuit}$ of a prestack should be, presumably the left Kan extension of $\mathrm{QCoh}^{\heartsuit}$ on affines? It will not matter later on since we consider flat quasicoherent sheaves only.

Now that we have constructed a functor that constructs homology theories on motivic spectra from flat formal groups, we tackle the question of representability, i.e. lifting this to motivic spectra, and whether this lift can be made symmetric monoidal.

Remark 12.16. The category $\mathrm{Mod}_{\mathrm{fg}}$ constructed above admits a symmetric mnoidal structure given by

$$(\mathscr{X},\mathscr{F})\otimes(\mathscr{Y},\mathscr{G})=(\mathscr{F}\times_{\mathscr{M}_{\mathrm{fg}}}\mathscr{Y};\mathscr{F}\boxtimes\mathscr{G}).$$

It is clear that the unit for this monoidal structure is $(\mathscr{M}_{\mathrm{fg}},\mathscr{O}_{\mathscr{M}_{\mathrm{fg}}})$. Further, we see that $\mathrm{Mod}_{\mathrm{fg}}^{\flat}$ is closed under binary tensor products, but does not contain the unit element.

This shows that we can only consider $\mathrm{Mod}_{\mathrm{fg}}^{\flat}$ as a non-unital symmetric monoidal category. We can either encode this as an operad and ask for "symmetric monoidal" functors to be maps of operads, or artificially add in a unit.

Definition 12.17. Let $\mathrm{Mod}_{\mathrm{fg}}^{\flat,+}$ denote the category $\mathrm{Mod}_{\mathrm{fg}}^{\flat}\cup\{(\mathscr{M}_{\mathrm{fg}},\mathscr{O})\}$ viewed as a symmetric monoidal full subcategory of $\mathrm{Mod}_{\mathrm{fg}}$.

Remark 12.18. The functor

$$\Phi_*\colon \mathrm{Mod}_{\mathrm{fg}}\to \mathrm{Fun}(\mathrm{MS}_S,\mathrm{Ab}^{\mathrm{Gr}})$$

defined by $\Phi_*(\mathscr{X},\mathscr{F})=\Phi_*\pi_*\mathscr{F}$ is lax symmetric monoidal, whence the target is equipped with the Day convolution monoidal structure.

Theorem 12.19 (Motivic Landweber Exact Fucntor Theorem). *Let S be a qcqs derived scheme, then there exists a unique symmetric monoidal coproduct-preserving functor Φ which is natural in S and fits in the commutative diagram*

$$\begin{tikzcd}
\mathrm{Mod}_{\mathrm{fg}}^{\flat,+} \arrow[r, "\Phi"] \arrow[d, hook] & \overline{\mathrm{h}}\mathrm{MS}_S^{\mathrm{lisse}} \arrow[d] \\
\mathrm{Mod}_{\mathrm{fg}}^{\flat} \arrow[r, "\Phi_*"] & \mathrm{Fun}(\mathrm{MS}_S,\mathrm{Ab}^{QS^0}).
\end{tikzcd}$$

Remark 12.20. The upper right vertex is a quotient of the homotopy category $\mathrm{hMS}_S^{\mathrm{lisse}}$ of lisse motivic spectra by the phantom maps. These are precisely the maps $f\colon E\to F$ of lisse motivic spectra such that for all compact lisse motivic spectra X (i.e. dualisable motivic spectra), with a map $u\colon X\to E$, the composite $f\circ u$ is nullhomotopic. These are in particular maps of lisse motivic spectra that induce the zero map on induces homology theories. The projection functor

$$\mathrm{MS}_S^{\mathrm{lisse}}\to\overline{\mathrm{h}}\mathrm{MS}_S^{\mathrm{lisse}}$$

is essentially surjective, full, conservative, symmetric monoidal, and coproduct-preserving.

Remark 12.21. Note that the right hand vertical functor is the obvious one, sending E to $E_*(-)$, but this admits a natural refinement to a $QS^0=\mathscr{K}(\mathbb{F}_1)$-graded object in abelian groups.

Example 12.22. Let $\widehat{\mathbb{G}}_m\colon \mathrm{Spec}(\mathbb{Z})\to\mathscr{M}_{\mathrm{fg}}$ denote the multiplicative formal group, which we previously showed satisfies the Landweber criterion, so that this lies in $\mathrm{Mod}_{\mathrm{fg}}^{\flat,+}$. Then

$$\Phi(\widehat{\mathbb{G}}_m)\simeq \mathrm{KGL}.$$

Proof. This follows from the previously established Conner–Floyd isomorphism. Indeed, we know that there is an isomorphism of cohomology theories

$$\mathrm{KGL}^*(-)\cong\mathrm{MGL}^*(-)\otimes_L\mathbb{Z}[\beta^{\pm}]\colon \mathrm{MS}_S^{\omega,\mathrm{op}}\to\mathrm{Ab}^{\mathrm{Gr}}.$$

Since dualisable object are compact in MS_S, we can convert this to an isomorphism of homology theories on dualisable objects, and note that both are filtered cocontinuous to extend this to lisse objects, i.e.

$$\mathrm{KGL}_*(-) \cong \mathrm{MGL}_*(-) \otimes_L \mathbb{Z}[\beta^\pm] \colon \mathrm{MS}_S^{\mathrm{lisse}} \to \mathrm{Ab}^{\mathrm{Gr}}.$$

The right hand of this isomorphism is clearly the homology theory associated to $\Phi(\widehat{\mathbb{G}}_m)$, so it suffices to note that KGL and MGL are both lisse, as was established in Recollection 1 to lift this to an equivalence in $\overline{\mathrm{h}}\mathrm{MS}_S^{\mathrm{lisse}}$ using the Yoneda lemma. □

Remark 12.23. the fact that Φ is symmetric monoidal tells us a lot. For example, it is then clear that

$$\Phi(\widehat{\mathbb{G}}_m \times \widehat{\mathbb{G}}_m) \simeq \mathrm{KGL} \otimes \mathrm{KGL},$$

so that $\widehat{\mathbb{G}}_m \times \widehat{\mathbb{G}}_m$ encodes co-operations in algebraic K-theory.

Let us now sketch the steps toward proving the motivic LEFT. First, we prove existence (and uniqueness) of Φ using representability.

Theorem 12.24 (Adams' homological Brown representability). *Let $\mathscr{C} \in \mathrm{CAlg}(\mathcal{P}\mathrm{r}_{\mathrm{St}}^{\mathrm{L},\omega})$ be a stable presentable symmetric monoidal compactly generated category such that $\mathscr{C}^\omega = \mathscr{C}^{\mathrm{du}}$. Then if $\mathscr{C}^\omega$ is countable, there is a commutative diagram*

$$\begin{tikzcd} \mathscr{C} \arrow[r] \arrow[d] & \mathrm{Fun}(\mathscr{C}, \mathrm{Ab}) \\ \overline{\mathrm{h}}\mathscr{C} \arrow[r] & \mathrm{Fun}^{\mathrm{hom}}(\mathscr{C}, \mathrm{Ab}), \arrow[u, hook] \end{tikzcd}$$

such that the bottom horizontal arrow is a strict symmetric monoidal equivalence. The top horizontal arrow is given by

$$E \mapsto \pi_0 \mathrm{Map}_{\mathscr{C}}(\mathbb{1}_{\mathscr{C}}, E \otimes -),$$

and the bottom right vertex consists of the full subcategory on homological functors.

Remark 12.25. The example to keep in mind is of course $\mathscr{C} = \mathrm{MS}_S^{\mathrm{lisse}}$, where $\mathscr{C}^\omega = \mathrm{MS}_S^{\mathrm{lisse},\omega} \simeq \mathrm{MS}_S^{\mathrm{du}}$ per construction. The condition that $\mathscr{C}^\omega$ be countable is satisfied when S is countable. The strategy is therefore to prove the motivic LEFT over a countable base such as $S = \mathrm{Spec}(\mathbb{Z})$ and to pull back to general bases from there. This latter operation is rather nontrivial.

To prove that the functor Φ in the motivic LEFT is symmetric monoidal, we can just apply a simple trick: let $M_1 \colon L \to R_1$ and $M_2 \colon L \to R_2$ be two formal group laws, then write

$$\begin{aligned} (\Phi(M_1) \otimes \Phi(M_2))_*(X) &\cong \Phi(M_1)_*(\Phi(M_2) \otimes X), \\ &\cong \mathrm{MGL}_*(\Phi(M_2) \otimes X) \otimes_L R_1, \\ &\cong \Phi(M_2)_*(\mathrm{MGL} \otimes X) \otimes_L R_1, \\ &\cong (\mathrm{MGL}_*(\mathrm{MGL}_\otimes X) \otimes_L R_2) \otimes_L R_1, \\ &\cong (\mathrm{MGL} \otimes \mathrm{MGL})_*(X) \otimes_{LB} (R_1 \otimes_L R_2), \\ &\cong \Phi(M_1 \boxtimes M_2)_*(X), \end{aligned}$$

where we used that the L-module structure on R_i came from two different copies of MGL_* to obtain the tensor product over LB in the fifth isomorphism.

13 Rational and étale motivic cohomology

Recall that for every $k \in \mathbb{Z}$ we have a map of $\mathbb{E}_\infty$-monoids

$$\mathrm{Pic} \to \mathrm{Pic}, \mathscr{L} \mapsto \mathscr{L}^{\otimes k}.$$

Therefore, we obtain an $\mathbb{E}_\infty$-ring map

$$\psi^k \colon \Sigma^\infty_{\mathbb{P}^1} \mathrm{Pic}_+ \to \Sigma^\infty_{\mathbb{P}^1} \mathrm{Pic}_+$$

in MS_S. Recall that the universal property (à la Snaith's theorem) of K-theory gave us an equivalence

$$\mathrm{KGL} \simeq \Sigma^\infty_{\mathbb{P}^1} \mathrm{Pic}_+[\beta^{-1}],$$

where $\beta \colon \mathbb{P}^1 \to \mathrm{Pic}_+$ classifies $\mathscr{O} - \mathscr{O}(-1)$. We readily compute that

$$\begin{aligned} \psi^k(\beta) &= \psi^k(1 - \mathscr{O}(-1)), \\ &= 1 - \psi^k(\mathscr{O}(-1)), \\ &= 1 - \mathscr{O}(-1)^{\otimes k}, \\ &= 1 - (1-\beta)^k \\ &\equiv_{\beta^2} k\beta. \end{aligned}$$

This gives us an equivalence

$$\psi^k(\beta) = k\beta \in \mathrm{KGL}^0(\mathbb{P}^1_S) \cong \mathrm{KGL}^0(S)[\beta]/\beta^2.$$

We conclude that this begets an $\mathbb{E}_\infty$-ring map

$$\Psi^k \colon \mathrm{KGL}[1/k] \to \mathrm{KGL}[1/k]$$

which is the k-th Adams operation.

Definition 13.1. [Beilinson's rational motivic cohomology] Let X be qcqs derived scheme and $n \in \mathbb{Z}$. Beilinson defines the rational motivic cohomology of X as

$$\mathrm{H}^*_{\mathrm{mot}}(X; \mathbb{Q}(n)) := \mathrm{K}_{2*-n}(X)^{(n)},$$

where the right hand side the the k^n-eigenspace of Ψ^k for some $k \notin \{0, \pm 1\}$.

Remark 13.2. The definition as above turns out to be independent of the choice of k.

The natural question is then whether this is representable by a motivic spectrum. We answer this in the affirmative.

Theorem 13.3 (Existence). *Beilinson's* $\mathrm{H}^*_{\mathrm{mot}}(;\mathbb{Q}(n))$ *is representable by an* $\mathbb{E}_\infty$*-ring spectrum* $\mathrm{H}\mathbb{Q} \in \mathrm{CAlg}(\mathrm{MS}_S)$ *which is furthermore an idempotent algebra and stable under base change in* S.

Remark 13.4. Recall that an algebra is idempotent if and only if the multiplication map $\mu \colon \mathrm{H}\mathbb{Q} \otimes \mathrm{H}\mathbb{Q} \to \mathrm{H}\mathbb{Q}$ is an equivalence. Equivalently, this says that the forgetful functor $\mathrm{Mod}(\mathrm{MS}_S; \mathrm{H}\mathbb{Q}) \to \mathrm{MS}_S$ is fully faithful, i.e. being an $\mathrm{H}\mathbb{Q}$-module is a *property*.

We can in fact characterise the full subcategory of $\mathrm{H}\mathbb{Q}$-modules. For this, note that $\mathrm{H}\mathbb{Q}$ is clearly a rational motivic spectrum purely by virtue of the fact that it represents a rational theory, so that we can equivalently just characterise $\mathrm{Mod}((\mathrm{MS}_S)_\mathbb{Q}; \mathrm{H}\mathbb{Q})$.

Theorem 13.5 (Characterisation of $\mathrm{H}\mathbb{Q}$-modules). *Let* $E \in (\mathrm{MS}_S)_\mathbb{Q}$ *be a rational motivic spectrum. The following are equivalent (the last two under some assumptions).*

1. *E lands in* $\mathrm{Mod}((\mathrm{MS}_S)_\mathbb{Q}; \mathrm{H}\mathbb{Q})$*, i.e. admits a (necessarily unique)* $\mathrm{H}\mathbb{Q}$*-module structure.*

2. *E is oriented.*

3. *The map $\langle -1\rangle \colon E \to E$ is homotopic to the identity.*

4. *The map $\eta \colon \Sigma_{\mathbb{P}^1} E \to E$ is nullhomotopic.*

5. *If 2 is invertible on the base S, then it is equivalent to E satisfying étale descent.*

6. *If the base scheme S contains a square root of -1, then it just true that any rational motivic spectrum is an $\mathrm{H}\mathbb{Q}$-module*

Remark 13.6. In the theorem statement above, the map $\langle -1\rangle$ is induced from the construction

$$a \in \mathscr{O}(S)^{\times}, \langle a\rangle := \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \colon \mathbb{P}^1_S \to \mathbb{P}^1_S$$

that gives rise to an endomorphism of the unit. The map η is the algebraic Hopf map $\eta \colon \mathbb{P}^1_S \to S^1$ defined similarly as in SH.

In fact, Beilinson's rational motivic cohomology can be recovered as the rational part of étale motivic cohomology, which arises as the associated graded on a filtered enhancement of Selmer K-theory.

Theorem 13.7 (Étale motivic cohomology). *There exists a filtered $\mathbb{E}_\infty$-algebra $(\Sigma^{\star}_{\mathbb{P}^1}\mathrm{kgl}^{\text{ét}}, \beta)$ in MS_S with colimit*

$$\mathrm{KGL}^{\text{ét}} = \mathrm{kgl}^{\text{ét}}[\beta^{-1}]$$

and associated graded[16]

$$\mathrm{H}\mathbb{Z}^{\text{ét}} = \mathrm{kgl}^{\text{ét}}/\beta.$$

These satisfy the following.

1. *All three are stable under base change.*

2. $\mathrm{KGL}^{\text{ét}}$ *represents Selmer* K*-theory.*

3. *The rationalisation $\mathrm{H}\mathbb{Z}^{\text{ét}}_{\mathbb{Q}} \simeq \mathrm{H}\mathbb{Q}$ represents Beilinson's rational motivic cohomology.*

4. *For any prime p, the completion $(\mathrm{H}\mathbb{Z}^{\text{ét}})^{\wedge}_p$ recovers the global syntomic cohomology of Bhatt–Lurie, i.e.*

   - *If p is invertible on S, then it recovers the étale cohomology of the generic fibre $\mathrm{R}\Gamma_{\text{ét}}(-;\mathbb{Z}_p(*))$.*
   - *If S is p-complete, then it recovers the syntomic cohomology of the associated p-adic formal scheme $\mathrm{R}\Gamma_{\mathrm{syn}}(-;\mathbb{Z}_p(*))$.*

5. $\mathrm{H}\mathbb{Z}^{\text{ét}}$ *satisfies étale descent*[17]*.*

6. $\mathrm{KGL}^{\text{ét}}$ *is the étale localisation of* KGL *in* MS_S*, i.e. its image under the left adjoint of the inclusion*

$$\mathrm{MS}^{\text{ét}}_S := \mathrm{Sp}_{\mathbb{P}^1}(\mathcal{S}\mathrm{hv}(\mathrm{Sm}_S;\mathrm{Sp})_{\text{ét,ebe}}) \hookrightarrow \mathrm{MS}_S.$$

7. *If S is Dedekind, so that motivic cohomology over S is well defined, then $\mathrm{H}\mathbb{Z}^{\text{ét}}$ is the étale localisation of $\mathrm{H}\mathbb{Z}$ in MS_S.*

8. *If S is Dedekind, then the terms $\Omega^{\infty-n}_{\mathbb{P}^1}\mathrm{H}\mathbb{Z}^{\text{ét}}$ represents the étale sheafification $L_{\text{ét}}z^n(-,\star)$ of Bloch's cycle complex.*

[16] one ought to view this as a graded object such that its graded parts recover e.g. the twists in global syntomic cohomology. However, using the projective bundle formula in KGL, we see that this can just be recovered from the weight bigrading.

[17] This can also be said about $\mathrm{KGL}^{\text{ét}}$, but this is clear since we have already identified it with Selmer K-theory.

Proof. Let us sketch the proof. First, note that we define $\mathrm{kgl}^{\text{ét}}$ to be the étale localisation of in MS_S. We see that the stability under base change follows from that of $\mathrm{H}\mathbb{Q}$ and $\mathrm{H}\mathbb{Z}_p^{\text{ét}}$ by a fracture square argument. It is not immediately clear that the gluing datum $(\mathrm{H}\mathbb{Z}_p^{\text{ét}})_{\mathbb{Q}}$ is stable under base change, but this follows from an argument using finite étale transfers in rationalised global syntomic cohomology.

Let us now assume that S is Dedekind, then it will suffice to construct an $\mathbb{E}_\infty$-map $\mathrm{H}\mathbb{Z} \to \mathrm{H}\mathbb{Z}^{\text{ét}}$. This is tricky, since we know that $\mathrm{H}\mathbb{Z}$ is stable under base change in SH, per construction, but not necessarily in MS. Therefore, we resort to afracture square again. Indeed, these agree rationally, and profinitely. To check that the gluing data agree, it therefore suffices to obtain a filler in the diagram

$$\begin{tikzcd} \mathrm{H}\mathbb{Q} \arrow[r] \arrow[dr] & (\prod_p \mathrm{H}\mathbb{Z}_p^\wedge)_{\mathbb{Q}} \arrow[d] \\ & (\prod_p (\mathrm{H}\mathbb{Z}^{\text{ét}})_p^\wedge)_{\mathbb{Q}}, \end{tikzcd}$$

but this is automatic since $\mathrm{H}\mathbb{Q}$ is idempotent so any such triangle commutes uniquely. □

13.1 Adams decomposition

If we now want to define rational motivic cohomology in terms of the Adams eigenspace decomposition of Beilinson, we see that this can not be done on a spetrum level. Indeed, the natural candidate for an eigenspace

$$\mathrm{fib}(\Psi^k - k^n \colon \mathrm{KGL}_{\mathbb{Q}} \to \mathrm{KGL}_{\mathbb{Q}})$$

is *not* a summand $\mathrm{KGL} \otimes \mathbb{Q}$.

Remark 13.8. This problem already shows up in usual homotopy theory. Indeed, there is an equivalence

$$\mathrm{fib}(\Psi^k - \mathrm{id} \colon \mathrm{KU}_{\mathbb{Q}} \to \mathrm{KU}_{\mathbb{Q}}) \simeq \mathbb{Q} \oplus \mathbb{Q}[-1].$$

It clearly suffices to take the connective cover of this object to obtain the correct summand $\mathbb{Q} \in \mathrm{CAlg}(\mathrm{Sp})$, and lax monoidality of the Whitehead cover will give this the correct $\mathbb{E}_\infty$-structure. However, we do not have access to a t-structure on MS_S.

Therefore, we want to rephrase the Adams decomposition on rational KGL in terms of an idempotent, and one such that the summand it picks out inherits the correct $\mathbb{E}_\infty$-structure. This calls for the motivic Landweber exact functor theorem. Recall that the functor

$$\Phi \colon \mathrm{Mod}_{\mathrm{fg}}^{\mathrm{b},+} \to \overline{\mathrm{h}}\mathrm{MS}_S^{\mathrm{lisse}}$$

is symmetric monoidal and sends $\widehat{\mathbb{G}}_m$ to KGL. In particular, we can recover the Čech nerve of the map

$$\widehat{\mathbb{G}}_m \colon \mathrm{Spec}(\mathbb{Z}) \to \mathscr{M}_{\mathrm{fg}}$$

since the target has affine diagonal so that every term in the nerve is affine and can be viewed as an object of $\mathrm{Mod}_{\mathrm{fg}}^{\mathrm{b},+}$. Furthermore, this lifts along the canonical $\mathbb{G}_m$-torsor on $\mathscr{M}_{\mathrm{fg}}$ to give us a pair of Čech nerves

$$\begin{tikzcd} \mathscr{M}_{\mathrm{fg}}^s \arrow[d] & \mathrm{Spec}\,\mathbb{Z}[\beta^\pm] \arrow[l] \arrow[d] & \mathrm{Spec}\,\Gamma_m \arrow[l, shift left] \arrow[l, shift right] \arrow[d] & \cdots \arrow[l] \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[d] \\ \mathscr{M}_{\mathrm{fg}} & \mathrm{Spec}\,\mathbb{Z} \arrow[l, "\mathbb{G}_m"'] & \mathrm{Spec}\,\mathbb{Z} \times_{\mathscr{M}_{\mathrm{fg}}} \mathrm{Spec}\,\mathbb{Z} \arrow[l, shift left] \arrow[l, shift right] & \cdots, \arrow[l] \arrow[l, shift left=2] \arrow[l, shift right=2] \end{tikzcd}$$

where the top simplicial diagram corresponds to a $\mathbb{Z}$-graded Hopf algebroid $(\mathbb{Z}[\beta^\pm], \Gamma_m)$ that presents its image in $\mathscr{M}_{\mathrm{fg}}$, i.e. the moduli stack $\mathscr{M}_{\mathrm{fg}}^m$ of multiplicative formal groups. On Φ, this simplicial diagramm gives rise to the Čech conerve $\mathrm{KGL}^{\otimes \bullet}$ of the unit map $\mathbb{1} \to \mathrm{KGL}$.

Remark 13.9. The ring Γ_m classifies strict isomorphisms of multiplicative formal group laws, and can be written more explicitly as

$$\Gamma_m = \mathbb{Z}[\beta^{\pm}] \otimes_L LB \otimes_L \mathbb{Z}[\beta^{\pm}]$$

along the left and right unit maps of LB. In fact, general formal group law theory tells us that Γ_m decomposes as a countable sum of copies of $\mathbb{Z}[\beta^{\pm}]$ as a $\mathbb{Z}[\beta^{\pm}]$-module.

Corollary 13.10. *The remark above allows us to make several analogous observations on the level of motivic spectra.*

1. *Since* Φ *preserves coproducts, we obtain an equivalence*

$$\mathrm{KGL} \otimes \mathrm{KGL} \simeq \bigoplus_{\mathbb{N}} \mathrm{KGL}$$

in $\overline{\mathrm{h}}\mathrm{MS}_S^{\mathrm{lisse}}$*, hence in* MS_S *of* KGL*-modules.*

2. *There is a coCartesian map of* $\mathbb{Z}$*-graded Hopf algebroids*

$$(\mathbb{Z}[\beta^{\pm}], \Gamma_m) \to (\mathrm{KGL}_*, \mathrm{KGL}_*\mathrm{KGL}),$$

where we recall that KGL *is just given by* $\mathrm{K}_0(S)[\beta^{\pm}]$ *by Bott periodicity.*

3. *In particular, the co-operations in* KGL *are given by*

$$\mathrm{KGL}_*\mathrm{KGL} \cong \mathrm{KGL}_* \otimes [\mathbb{Z}[\beta^{\pm}]\Gamma_m.$$

4. *We can further iterpret this as saying that the Hopf algebroid* $(\mathrm{KGL}_*, \mathrm{KGL}_*\mathrm{KGL})$ *of co-operations in algebraic* K*-theory is algebraically determined by the coaction map*[18]

$$\mathrm{KGL}_* \to \mathrm{KGL}_* \otimes_{\mathbb{Z}[\beta^{\pm}]} \Gamma_m.$$

To obtain the ring of operations in K-theory, where Adams operations naturally live, we will have to dualise the process above. Fortunately, we saw that $\mathrm{KGL} \otimes \mathrm{KGL}$ is a free KGL-module on countably many generators, whence we obtain

$$\mathrm{KGL}^*\mathrm{KGL} \cong \mathrm{Hom}_{\mathrm{KGL}_*}(\mathrm{KGL}_*\mathrm{KGL}, \mathrm{KGL}_*).$$

Remark 13.11. In this expression, the left hand side is understood to have an associative ring structure coming from composition of self-maps KGL, i.e. composition of operations, while the right hand side also has an associative ring structure coming from the coassociative comultiplication in $\mathrm{KGL}_*\mathrm{KGL}$ under duality. These can be seen to agree by a similar argument as in the case of Adams type homology theories on spectra. Indeed, this comes from the generl observation that if (A, Γ) is a (flat) Hopf algebroid, then the A-linear dual (the A-module structure on Γ being taken bia the left unit map) $\Gamma^\vee = \mathrm{Hom}_A(\Gamma, A)$ admits the structure of an associative ring in (A, A)-bimodules.

Example 13.12. This can be made more explicit. When $(A, \Gamma) = (\mathbb{Z}[\beta^{\pm}], \Gamma_m)$ is the Hopf algebroid presenting the moduli of multiplicative formal group laws, then we can write

$$\Gamma_m^\vee \cong \varprojlim(\cdots \xrightarrow{\omega} \mathbb{Z}[\![x]\!] \xrightarrow{\omega} \mathbb{Z}[\![x]\!])[\beta^{\pm}],$$

as a limit of abelian groups, where the map ω is defined on a power series f by

$$\omega(f) = (1-x)\frac{\mathrm{d}f}{\mathrm{d}x}.$$

The terms $\mathbb{Z}[\![x]\!]$ admit the structure of associative topological rings (with multiplication denoted $\bullet$ and topology given by the x-adic topology) by the rule

$$(1-x)^{-k} \bullet (1-x)^{-\ell} = (1-x)^{-k-\ell}.$$

[18]Note that this coaction is a priri terribly complicated.

Remark 13.13. It is unreasonable to ask for $\Gamma_m^\vee$ to be commutative, as a ring of operations under composition. It turns out that all the of the terms $(\mathbb{Z}[\![x]\!], \bullet)$ are commutative (as can be clearly seen from the definition of $\bullet$), but this is a coincidence and not a feature. In fact, the limit operation destroys this commutativity as in $\Gamma_m^\vee$ we have the identity

$$\beta^{-1} a \beta = \omega(a)$$

for all $a \in \Gamma_m^\vee$.

Remark 13.14. Furthermore, while $\Gamma_m^\vee$ is an associative algebra in $\mathbb{Z}[\beta^\pm]$-bimodules, it is not a $\mathbb{Z}[\beta^\pm]$-algebra since the element β is not central as we illustrated above. This claim is erroneously made in the literature (e.g. Naumann–Spitzweck–Østvær).

Corollary 13.15. *there is an isomorphism of associative rings*

$$\mathrm{KGL}^*\mathrm{KGL} \cong \mathrm{KGL}^*(S) \widehat{\otimes}_{\mathbb{Z}[\beta^\pm]} \Gamma_m \vee .$$

Remark 13.16. Following the remark above, beware that the ring structure on the right hand side is not automatic, since $\Gamma_m^\vee$ does not admit the structure of a $\mathbb{Z}[\beta^\pm]$-algebra. The ring structure instead comes from the coaction of $(\mathbb{Z}[\beta^\pm], \Gamma_m)$ on KGL_*.

Example 13.17. If the base is such that $\mathrm{K}_0(S) = \mathbb{Z}$, e.g. for a field or PID, then for any subring $\Lambda \subset \mathbb{Q}$ this expression simplifies to

$$\mathrm{KGL}^*_\Lambda \mathrm{KGL} \cong \Lambda \widehat{\otimes} \Gamma_m^\vee.$$

For example, if $\Lambda = \mathbb{Z}[1/k]$, then the operation $\Psi^k \in \mathrm{KGL}^*_{\mathbb{Z}[1/k]}\mathrm{KGL}$ corresponds to the compatible family of power series

$$(k^{-n}(1-x)^{-k})_n \in \varprojlim_{\omega} \mathbb{Z}[1/k][\![x]\!].$$

indeed, it is readily checked that $\omega(k^{-n}(1-x)^{-k}) = k^{1-n}(1-x)^{-k}$.

Consider $\Lambda = \mathbb{Q}$ in the example above, and note that

$$\widehat{\mathbb{G}}_a \colon \operatorname{Spec}(\mathbb{Z}) \to \mathscr{M}_{\mathrm{fg}}$$

has image the moduli $\mathscr{M}_{\mathrm{fg}}^a$ of additive formal groups presented by the Hopf algebroid $(\mathbb{Z}[\beta^\pm], \Gamma_a)$. Then Γ_a is rather complicated, indeed, $\Gamma_a \otimes \mathbb{F}_p$ recovers the dual Steenrod algebra $\mathscr{A}_p^\vee$ as the ring of automorphisms of the universal additive formal group in characteristic p. However, there is a rational isomorphism

$$\widehat{\mathbb{G}}_{m,\mathbb{Q}} \cong \widehat{\mathbb{G}}_{a,\mathbb{Q}}$$

given by the logarithm and exponential. These assemble to an equivalence of (rationalised) stacks

$$\mathscr{M}^a_{\mathrm{fg},\mathbb{Q}} \cong \mathscr{M}^m_{\mathrm{fg},\mathbb{Q}}$$

and in particular an isomorphism of rings

$$\mathbb{Q}\widehat{\otimes}\Gamma_m^\vee \cong \mathbb{Q}\widehat{\otimes}\Gamma_a^\vee.$$

Now the right hand side can be described more easily as the ring $\mathbb{Q}^{\mathbb{Z}}[\beta^\pm]$ with the ring structure on $\mathbb{Q}^{\mathbb{Z}}$ given by pointwise multiplication and β a non-central element satisfying the conjugation relation

$$\beta^{-1} a \beta = \mathrm{sh}(a),$$

sh denoting the shift of a sequence in $\mathbb{Q}^{\mathbb{Z}}$. Under this isomorphism, $\Psi^k \in \mathbb{Q}\widehat{\otimes}\Gamma_m^\vee$ corresponds to the family $(k^n)_n$ in $\mathbb{Q}\widehat{\otimes}\Gamma_a^\vee \cong \mathbb{Q}^{\mathbb{Z}}[\beta^{-1}]$.

Definition 13.18. for every $n \in \mathbb{Z}$, let $e_n \in \pi_0\mathrm{End}(\mathrm{KGL}_{\mathbb{Q}})$ denote the idempotent corresponding to the n-th basis vector in $\mathbb{Q}^{\mathbb{Z}}$. then we set $\mathrm{KGL}_{\mathbb{Q}}^{(n)}$ to be the image of e_n, viewed as a summand of $\mathrm{KGL}_{\mathbb{Q}}$. Further, write $\mathrm{H}\mathbb{Q} = \mathrm{KGL}_{\mathbb{Q}}(0)$.

Proposition 13.19.

1. *The idempotents are complete and orthonormal* $\mathrm{KGL}_{\mathbb{Q}}$ *, i.e. there is an isomorphism*

$$\mathrm{KGL}_{\mathbb{Q}} \cong \bigoplus_{n\in\mathbb{Z}} \mathrm{KGL}_{\mathbb{Q}}^{(n)}.$$

2. *The Bott map* β *induces an equivalence*

$$\Sigma_{\mathbb{P}^1}\mathrm{KGL}_{\mathbb{Q}}^{(n)} \xrightarrow{\sim} \mathrm{KGL}_{\mathbb{Q}}^{(n+1)}.$$

3. *The inclusion* $\mathrm{KGL}_{\mathbb{Q}}^{(n)} \to \mathrm{KGL}_{\mathbb{Q}}$ *equalises the maps* (Ψ^k, k^n).

Proof.

1. This is an immediate consequence of the corresponding fact about the e_n as idempotents in $\mathbb{Q}^{\mathbb{Z}}[\beta^{\pm}]$, indeed, they are such that $e_i e_j = 0$ for $i \neq j$ and the power series $\sum_n e_n$ converges to one.
2. In $\mathbb{Q}^{\mathbb{Z}}[\beta^{\pm}]$ we have the conjugation relation $\beta^{-1}e_n\beta = \mathrm{sh}(e_n) = e_{n-1}$.
3. In $\mathbb{Q}^{\mathbb{Z}}[\beta^{\pm}]$ we have that $\Psi^k e_n = k^n e_n$ since we identified $\Psi^k = (k^n)_n$.

□

13.2 Interpretation using the LEFT

It turns out that we may as well have constructed these $\mathrm{KGL}_{\mathbb{Q}}^{(n)}$ using the motivic LEFT. However, this description does not give us the desired interpretation in terms of Adamss eigenspaces. for this construciton, consider the (now Landweber flat as we are working over a rational base) map

$$\begin{tikzcd} \mathrm{Spec}(\mathbb{Q}) \arrow[r, "\widehat{\mathbb{G}}_a"] \arrow[dr, "\pi"'] & \mathscr{M}_{\mathrm{fg}} \\ & \mathrm{Spec}(\mathbb{Q})/\mathbb{G}_m \arrow[u, "\widehat{\mathbb{G}}_a^{\mathrm{gr}}"'] \end{tikzcd}$$

classifying $\widehat{\mathbb{G}}_a$ over $\mathbb{Q}$ as well as its unique, factorisation through $\mathrm{Spec}(\mathbb{Q})/\mathbb{G}_m$ by virtue of the fact that the dualising line of the additive formal group is trivial whence it is uniquely coordinatisable. The LEFT then gives us an equivalence

$$\Phi(\widehat{\mathbb{G}}_a, \mathscr{O}_{\mathbb{Q}}) \simeq \mathrm{KGL}_{\mathbb{Q}},$$

and we can further identify the left hand side with its pushforward

$$\Phi(\widehat{\mathbb{G}}_a^{\mathrm{gr}}, \pi_*\mathscr{O}_{\mathbb{Q}}) \simeq \mathrm{KGL}_{\mathbb{Q}}.$$

Now the pushforward π from $\mathrm{Spec}(\mathbb{Q})$ to $\mathrm{Spec}(\mathbb{Q})/\mathbb{G}_m$ is such that on the structure sheaf

$$\pi_*\mathbb{Q} \cong \bigoplus_n \mathbb{Q}(n)$$

we obtain a copy of the structure sheaf in every grading degree. Therefore, we can define

$$\mathrm{KGL}_{\mathbb{Q}}^{(n)} = \Phi(\widehat{\mathbb{G}}_a^{\mathrm{gr}}, \mathbb{Q}(n)).$$

13.3 Multiplicative structure and idempotence

Let us now end by describing the $\mathbb{E}_\infty$-ring structure on $\mathrm{H}\mathbb{Q} = \mathrm{KGL}_{\mathbb{Q}}^{(0)}$ and showing that it is idempotent. since being idempotent is a property of an $\mathbb{E}_0$-algebra over the unit of MS_S, and every idempotent $\mathbb{E}_0$-algebrauniquely lifts to (an equally idempotent) $\mathbb{E}_\infty$-algebra, it suffices to prove the former. For this, we use the description of $\mathrm{H}\mathbb{Q}$ in terms of the LEFt. Indeed, Φ is symmetric monoidal, so it suffices to check that the $\mathbb{E}_0$-algebra

$$(\widehat{\mathbb{G}}_a^{\mathrm{gr}}, \mathbb{Q}(0)) \in \mathrm{Mod}_{\mathrm{fg}}^{\mathrm{b},+}$$

is idempotent. This is immediate, since its tensor product with itself is given by

$$\mathrm{Spec}(\mathbb{Q})/\mathbb{G}_m \times_{\mathscr{M}_{\mathrm{fg}}} \mathrm{Spec}(\mathbb{Q})/\mathbb{G}_m \xleftarrow{\Delta} \mathrm{Spec}(\mathbb{Q})/\mathbb{G}_m,$$

and since $\mathscr{M}_{\mathrm{fg}}$ has affine diagonal (affine as $\mathbb{G}_m$-stacks), we see that this reduces to checking that the diagonal map Δ above is actually an equivalence. However, we have presentations for both of these stacks, so it suffices to check that the map

$$\mathbb{Q} \otimes_{L,\eta_R} LB \otimes_{\eta_L,L} \mathbb{Q} \to \mathbb{Q}$$

classifying the identity automorphism of $\widehat{\mathbb{G}}_a$ over $\mathbb{Q}$ is an isomorphism, but this can be readily checked.

Remark 13.20. For clarity, we have written the maps $L \to LB$ involved in the formation of the tensor products. It is clear that if we took both maps to be η_L or η_R, we would end up with a polynomial algebra over $\mathbb{Q}$, which is not what we want!

By the strict monoidality of Φ, we conclude that the map

$$(\mathbb{1} \to \mathrm{H}\mathbb{Q}) \otimes \mathrm{H}\mathbb{Q}$$

obtained by applying Φ to the isomorphism $(\mathrm{Spec}(\mathbb{Q})/\mathbb{G}_m \to \mathscr{M}_{\mathrm{fg}}) \boxtimes \mathrm{Spec}(\mathbb{Q})/\mathbb{G}_m$ in $\mathrm{Mod}_{\mathrm{fg}}^{\mathrm{b},+}$ is an isomorphism in $\overline{\mathrm{h}}\mathrm{MS}_S^{\mathrm{lisse}}$, hence an equivalence in MS_S, and we are done.