Deformations of Stable Homotopy Theory

Masterarbeit der Mathematik

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Abstract

We provide an overview of some recent work on the theory of deformations of stable homotopy theories, as exemplified by the construction of synthetic spectra. We set up the ∞-categorical machinery to define a one-parameter deformation of homotopy theories, in the form of stable presentably symmetric monoidal ∞-categories. This notion of a deformation, inspired by observations from spectral algebraic geometry, is related to other (equivalent) definitions of a deformation appearing in the literature such as [BHS20]. In particular, we show how to determine the generic and special fibres of a deformation using a version of the Schwede–Shipley theorem introduced in [Heg18]. Within this notion of deformations, filtered spectra arise as the universal deformation, and we analyse their categorical properties, as well as prevalence in the theory of deformations of homotopy theories. The primary example of in this work is that of synthetic spectra as developed in [Pst18]. We elaborate on past work of Burklund–Hahn–Senger on showing that synthetic spectra admit the structure of a deformation, and work out some explicit identifications that can be made in this example, culminating in an explicit filtered model of a certain subcategory of synthetic spectra. The deficit between synthetic spectra and this subcategory is quantified using the theory of symmetric monoidal recollements of stable ∞-categories.

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Berlin, 22 Aug 2022 — Lucas Piessevaux

L.P.

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1 Introduction

A variety of modern advances on the Adams spectral sequence have taken place within the realm of synthetic spectra. This flavour of homotopy theory, developed in [Pst18] and based on an Adams-type homology theory, can be thought of as a certain deformation of the ∞-category of spectra Sp. Further, the computational programme of Isaksen et al. in approaching the stable homotopy groups of spheres using $\mathbb{C}$-motivic homotopy theory turns out to admit an interpretation as a deformation as well. Both of these examples, by work of [BHS20] and [Ghe+18] respectively, admit filtered models. That is, descriptions of these deformed homotopy theories as modules in filtered spectra. Unifying this drive towards filtered spectra is the folklore result that these deformations are in a sense parametrised by the spectral stack $\mathbb{A}^1/\mathbb{G}_m$, and that filtered spectra arise naturally as quasicoherent sheaves on this stack. This is made rigorous using techniques from spectral algebraic geometry in [Mou21].

Inspired by this, we aim to give a definition of what a deformation of homotopy theories should be, based on geometric intuition from op. cit. In fact, filtered spectra appear to play a key role in this definition. However, the word *deformation of homotopy theories* has been floating around in the literature for a while now, to varying levels of concreteness. Therefore, we focus on comparing our geometric definition of deformations with other examples found in the literature. Most importantly, the notion of a deformation of homotopy theories proposed in Appendix C of [BHS20]. As expected, these two models are equivalent, and one can choose either of them to work with. In fact, we stress the importance of the identification of a certain operator τ present in every deformed homotopy theory, called the *thread operator*. This operator shows up naturally in the construction of synthetic spectra and p-complete $\mathbb{C}$-motivic spectra, where it controls the deformation structure.

Since synthetic spectra admit a reasonably simple definition, we have opted to focus on these as the main example in this work. Especially since their deformation structure and its effects on computations in synthetic spectra are well documented and have played key roles in modern developments, cf. [BHS19]. However, there are many more deformations out there, such as the p-complete $\mathbb{C}$-motivic example, genuine C_2-equivariant spectra, etc. The hope of the author is that the framework established here can easily allow for new deformations of homotopy theories to be constructed and described, either in concrete definitions using families of dualisable objects and symmetric monoidal left adjoints, or even arising from spectral algebraic geometry–the two are equivalent.

The organisation of this thesis is as follows: First, we lay down the prerequisites (Section 4) of axiomatic stable homotopy theory, using stable presentably symmetric monoidal ∞-categories as our objects of interest. We assume that the reader is familiar with higher category theory up to the level that they are comfortable with the construction of ∞-operads, and we take this as our starting point. The contents of this section are not intended to be original.

Next, we give an in depth analysis of the ∞-category of filtered spectra in Section 5. As stated earlier, this plays an important role in the theory of deformations, and plays a central role in many computational applications. In particular, we study filtered spectra as filtrations of spectra, as well as geometric objects: quasicoherent sheaves on $\mathbb{A}^1/\mathbb{G}_m$. Finally, we give a description of a certain subcategory of filtered spectra in terms of cochain complexes in spectra due to [Ari21]. The latter example is used as an example of a deformation of homotopy theories, but its primary use–for defining a new t-structure on filtered spectra–goes beyond the scope of this work.

In Section 6 we recall the definition and some characterisations of recollements of stable ∞-categories. In particular, we refine to symmetric monoidal recollements using work of [Sha21]. These recollements will show up naturally in the study of deformations, arising from a recollement on quasicoherent sheaves associated to a closed-open decomposition of a spectral stack. In particular, they will end up allowing us to decompose deformed homotopy theories into τ-complete and τ-invertible parts, with both of these subcategories admitting more explicit descriptions in many cases.

Section 7 contains the theoretical meat of this work, including a recap of the geometry of the stack $\mathbb{A}^1/\mathbb{G}_m$ due to [Mou21]. Most importantly, it is in this section that we are able to give our definition of what a deformation of homotopy theories should be. We immediately compare our definition with the aforementioned definition from [BHS20] and show that they are equivalent. The main result of this section and this thesis is the identification of the generic and special fibres in a deformation of homotopy theories. Indeed, the folklore definition of a deformation consists of some span of homotopy theories with the two fibres lying under the deformed homotopy theory. We can now make this rigorous, using our description of filtered spectra and tensoring up along the definition of deformations.

The following Section 8 is a recap of the work in [Pst18] and [BHS19] on synthetic spectra, their homotopy theory, and different notions of completion internal to this ∞-category. In this rather technical section, we set up all of the work that will make the description of synthetic spectra as a deformation rather simple. Once again, we do not claim any originality for the results in this section

Finally, in Section 9, we assemble the work on describing synthetic spectra with the characterisations of deformations from Section 7 to work out our primary example of a deformation. This will be rather immediate, since the past two sections have been geared towards this application. Along with a comment on cellularity conditions for synthetic spectra, the main result is then the filtered model for τ-complete synthetic spectra. This result appears in [BHS20], and we use this final section to work out the equivalence given there, using our previous analysis of synthetic homotopy theory and deformations of homotopy theories. Finally, to work out the deficit between the τ-complete synthetic spectra and the total ∞-category, we use our section on recollements to reconstruct the entire ∞-category as a right lax limit of this filtered model and the ∞-category of spectra itself.

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3 Conventions and Notation

Unless stated otherwise, the word *∞-category* refers to an $(\infty, 1)$-category, for which we use Lurie and Joyal's formalism of quasicategories, using [Lur09] and [Lur17] as the primary references. Following this, we establish the following notational convention:

lim	(homotopy) Limit
colim	(homotopy) Colimit
$\mathcal{S}$	∞-category of spaces
Sp	∞-category of spectra
map	Mapping space in an ∞-category
Map	Mapping spectrum in an ∞-category
$\mathcal{P}(\mathcal{C})$	∞-category of presheaves of spaces on an ∞-category $\mathcal{C}$
$\mathcal{S}\mathrm{hv}$	∞-category of sheaves of spaces on a site $\mathcal{C}$
よ	Yoneda embedding
$\mathcal{F}\mathrm{un}$	∞-category of functors
$\mathbb{F}_*$	(Nerve of) the 1-category of finite pointed sets
$\mathcal{C}^{\otimes}$	∞-operad associated to a symmetric monoidal ∞-category $\mathcal{C}$
$\mathrm{Mod}(\mathcal{C}, R)$	∞-category of (right) modules over the $\mathbb{E}_n$-algebra R in $\mathcal{C}$.

4 Prerequisites

The goal of this section is to set up all the prerequisites necessary to be able to make statements about stable presentably symmetric monoidal ∞-categories, also referred to as stable homotopy theories or noncommutative stacks in this work. In particular, we will set up the notion of presentability, some constructions of symmetric monoidal structures on presentable ∞-categories and functor ∞-categories, as well as the notion of stability. The latter will be constructed internal to presentable ∞-categories, where it admits an elegant description in terms of smashing localisations. This section is meant as a recap of some common categorical machinery from homotopy theory, and is not meant to be a exhaustive or original discussion; the primary sources will be cited in each section.

4.1 Day convolution

Many ∞-categories arising in this work will admit a natural description as a functor ∞-category, or perhaps some full subcategory of the latter. In particular, this holds for ∞-categories of diagrams, (pre)sheaf ∞-categories, etc. Given two ∞-categories $\mathcal{C}$ and $\mathcal{D}$ with $\mathcal{D}$ cocomplete, it is then natural to ask what kind of symmetric monoidal structures can be put on the functor ∞-category $\mathcal{F}\mathrm{un}(\mathcal{C}, \mathcal{D})$. In fact, there is a natural choice for this monoidal structure called the Day convolution. This has the universal property that an $\mathbb{E}_\infty$-monoid for the Day convolution in $\mathcal{F}\mathrm{un}(\mathcal{C}, \mathcal{D})$ corresponds to the datum of a lax monoidal functor from $\mathcal{C}$ to $\mathcal{D}$. In fact, in the case that $\mathcal{D} = \mathcal{S}$ is the ∞-category of spaces, we obtain a nice interaction with the theory of presheaves: The Day convolution structure on $\mathcal{F}\mathrm{un}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ is universal with the property that the Yoneda embedding

$$よ : \mathcal{C} \to \mathcal{F}\mathrm{un}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$$

is symmetric monoidal. The extension of the Day convolution to a complete ∞-operadic description for ∞-categories described here is due to Glasman in [Gla16]. The goal of this section is to describe this construction, give some sketches of the corresponding proofs, and cite the main universal properties.

The setup for this construction involves two ∞-categories $\mathcal{C}$ and $\mathcal{D}$, the latter having all colimits. We equip them with a symmetric monoidal structure, encoded by coCartesian fibrations

$$p_{\mathcal{C}} : \mathcal{C}^{\otimes} \to \mathbb{F}_*, \qquad\qquad p_{\mathcal{D}} : \mathcal{D}^{\otimes} \to \mathbb{F}_*.$$

The first step towards defining a symmetric monoidal ∞-category of functors from $\mathcal{C}$ to $\mathcal{D}$ is to define a coCartesian fibration over $\mathbb{F}_*$ encoding this. In fact, one can work backwards from the universal properties of the Day convolution cited above (and with precedent in the 1-categorical setting, due to Day). In particular, since we want commutative algebras in $\mathcal{F}\mathrm{un}(\mathcal{C}, \mathcal{D})$ to be lax symmetric monoidal functors from $\mathcal{C}$ to $\mathcal{D}$, and one can identify the former with sections of its structure fibration down to $\mathbb{F}_*$, let us define a simplicial set $\widetilde{\mathcal{F}\mathrm{un}(\mathcal{C}, \mathcal{D})}^{\otimes}$ by

$$\mathcal{F}\mathrm{un}_{\mathbb{F}_*}(K, \widetilde{\mathcal{F}\mathrm{un}(\mathcal{C}, \mathcal{D})}^{\otimes}) \simeq \mathcal{F}\mathrm{un}_{\mathbb{F}_*}(\mathcal{C}^{\otimes}_k, \mathcal{D}^{\otimes}_k).$$

In this notation, $k : K \to \mathbb{F}_*$ is some map of simplicial sets, and the fibres $\mathcal{C}^{\otimes}_k$ are defined as

$$\mathcal{C}^{\otimes}_k := \mathcal{C} \times_{p_{\mathcal{C}}, \mathbb{F}_*, k} K,$$

and similarly for $\mathcal{D}^{\otimes}_k$. It then suffices to show that the simplicial set over $\mathbb{F}_*$ defined in this way actually defines a symmetric monoidal ∞-category to obtain an explicit construction of the Day convolution. However, this is a little more subtle. Indeed, we have not incorporated the Segal condition of $\mathcal{C}$ and $\mathcal{D}$, which tells us that there are decompositions

$$\mathcal{C}^{\otimes}_{\langle n \rangle} \simeq \mathcal{C}^n,$$

and it turns out that $\widetilde{\mathcal{F}\mathrm{un}(\mathcal{C}, \mathcal{D})}^{\otimes}$ as defined above is only a locally coCartesian fibration over $\mathbb{F}_*$. Indeed, this construction works for all coCartesian fibrations, so one can predict that the right answer actually uses that $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$ are symmetric monoidal ∞-categories. We see that there are a few more gaps to be filled in, and it will be necessary to restrict to a subsimplicial set of the latter to make up for these.

We omit the proof that the natural map

$$\widetilde{\mathcal{F}\mathrm{un}(\mathcal{C},\mathcal{D})}^{\otimes} \to \mathbb{F}_*$$

is a locally coCartesian fibration, and instead refer to [Gla16].

Now, let us make the statement above more rigorous. We wanted the construction of the ∞-operad encoding the Day convolution symmetric monoidal structure to encode the product decompositions of ∞-categories. Let us therefore begin by elaborating on this.

Recall that the product decomposition of a symmetric monoidal ∞-category

$$\mathcal{C}^{\otimes}_{\langle n\rangle} \simeq \mathcal{C}^n$$

is induced by the pushforward along the inert morphisms

$$\rho_i : \langle n\rangle \to \langle 1\rangle$$

picking out the element $i \in n$ as the preimage of 1. The product of these pushforwards on fibres is precisely what induces the decomposition above. First, let us generalise $\langle 1\rangle$ to a more general $\langle m\rangle$, for $m \geq 1$. Then a morphism

$$f : \langle n\rangle \to \langle m\rangle$$

is determined by the preimages of every non-basepoint element of $\langle m\rangle$, i.e. the restriction

$$f\mid_{f^1(\{i\})}: f^{-1}(\{i\}) \to \{i\} \subset \langle m\rangle,$$

equivalently encoded by an active morphism

$$f_i : f^{-1}(\{i\}) \to \langle 1\rangle \cong \{*, i\}.$$

The remaining information in f is then simply the preimage of the basepoint, i.e. the pointed finite set encoding what is thrown away. We conclude that there is a decomposition

$$\mathbb{F}_{*/\langle m\rangle} \simeq (\mathbb{F}^{\mathrm{act}}_{*/\langle 1\rangle})^m \otimes \mathbb{F}_*; f \mapsto ((f_i)^m_{i=1}, f^{-1}(*)).$$

In particular, a symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$ decomposes as

$$\mathcal{C}^{\otimes} \times_{\mathbb{F}_*} \mathbb{F}_{*/\langle m\rangle} \simeq (\mathcal{C}^{\otimes} \times_{\mathbb{F}_*} \mathbb{F}^{\mathrm{act}}_{*/\langle 1\rangle})^m \times \mathcal{C}^{\otimes}.$$

Setting $m = 0, 1$ recovers the usual product decomposition. The upshot of this description is that it captures decompositions along edges as well. Indeed, if we let $f : \langle n\rangle \to \langle m\rangle$ be a morphism in $\mathbb{F}_*$, viewed alternatively as an element in the fibre over $\langle n\rangle$ of $\mathbb{F}_{*/\langle m\rangle} \to \mathbb{F}_*$, or a morphism $f : \Delta^1 \to \mathbb{F}_*$, allowing us to construct

$$\mathcal{C}^{\otimes}_f = \mathcal{C}^{\otimes} \times_{p_{\mathcal{C}},\mathbb{F}_*,f} \Delta^1$$

Now note that the right hand side of this expression is also the fibre over $f \in \mathbb{F}_{*/\langle m\rangle}$ of $\mathcal{C} \times_{\mathbb{F}_*} \mathbb{F}_{*/\langle m\rangle}$, so that one can insert this into the product decomposition (taking fibres over f on both sides) to obtain

$$\begin{aligned}
\mathcal{C}^{\otimes}_f &\simeq (\mathcal{C}^{\otimes} \times_{\mathbb{F}_*} \mathbb{F}_{*/\langle m\rangle}) \times_{\pi_2,\mathbb{F}_{*/\langle m\rangle},f} \Delta^0, \\
&\simeq ((\mathcal{C}^{\otimes} \times_{\mathbb{F}_*} \mathbb{F}^{\mathrm{act}}_{*/\langle 1\rangle}) \times_{\pi_2,\mathbb{F}^{\mathrm{act}}_{*/\langle 1\rangle},f} \Delta^0)^m \times \mathcal{C} \times_{p_{\mathcal{C}},\mathbb{F}_*,f^{-1}(*)\to\langle 0\rangle} \Delta^1, \\
&\simeq \prod_{i=1}^m \mathcal{C}^{\otimes}_{\mu_{f^{-1}(i)}} \times \mathcal{C}^{\otimes}_{c_{f^{-1}(*)}},
\end{aligned}$$

where $c_{f^{-1}(*)}$ is the unique morphism $f^{-1}(*) \to \langle 0\rangle$ that shows up in the second line when one takes the preimage over the basepoint. The map $\mu_{f^{-1}(i)}$ is the unique active map to $\langle 1\rangle$ sending all non-basepoint elements to 1.

We can now define the symmetric monoidal ∞-category $\mathcal{F}\mathrm{un}(\mathcal{C},\mathcal{D})^{\otimes}$ with the Day convolution structure as the subsimplicial set of $\widetilde{\mathcal{F}\mathrm{un}(\mathcal{C},\mathcal{D})}^{\otimes}$ on vertices and edges that do have a product decomposition, i.e. we want

$$\begin{aligned}\mathcal{F}\mathrm{un}(\mathcal{C},\mathcal{D})^{\otimes}_{\langle n\rangle}([0]) =&\mathcal{F}\mathrm{un}(\mathcal{C},\mathcal{D})^{n}\\ &\hookrightarrow \mathcal{F}\mathrm{un}(\mathcal{C}^{n},\mathcal{D}^{n}) \simeq \mathcal{F}\mathrm{un}(\mathcal{C}^{\otimes}_{\langle n\rangle},\mathcal{D}^{\otimes}_{\langle n\rangle}) \simeq \widetilde{\mathcal{F}\mathrm{un}(\mathcal{C},\mathcal{D})}^{\otimes}{}_{\langle n\rangle}([0]),\\ \mathcal{F}\mathrm{un}(\mathcal{C},\mathcal{D})^{\otimes}_{f:\langle n\rangle\to\langle m\rangle}([1]) =& \prod_{i=1}^{m}\mathcal{F}\mathrm{un}_{\Delta^1}(\mathcal{C}^{\otimes}_{\mu f^{-1}(i)},\mathcal{D}^{\otimes}_{\mu f^{-1}(i)})\times \mathcal{F}\mathrm{un}^{1}_{\Delta}(\mathcal{C}^{\otimes}_{c_{f^{-1}(*)}},\mathcal{D}^{\otimes}_{c_{f^{-1}(*)}})\\ &\hookrightarrow \mathcal{F}\mathrm{un}^{1}_{\Delta}(\mathcal{C}^{\otimes}_{f},\mathcal{D}^{\otimes}_{f}) \simeq \widetilde{\mathcal{F}\mathrm{un}(\mathcal{C},\mathcal{D})}^{\otimes}{}_{f}([1]).\end{aligned}$$

It is clear from these expressions that this really is the subsimplicial set on functors respecting the product decomposition, and from the expression for the vertices, one sees that the underlying ∞-category of this symmetric monoidal ∞-category is precisely $\mathcal{F}\mathrm{un}(\mathcal{C},\mathcal{D})$.

Assuming that the tensor product of $\mathcal{D}$ commutes with colimits, one shows (cf. [Gla16]) that this object defines a full subcategory, and is a coCartesian fibration over $\mathbb{F}_*$, thus giving rise to the symmetric monoidal ∞-category $\mathcal{F}\mathrm{un}(\mathcal{C},\mathcal{D})$ of functors with the Day convolution as required.

The proof of the universal properties w.r.t lax monoidal functors and the symmetric monoidal Yoneda embedding are proven in op. cit. as Proposition 2.12 and Section 3.

Finally, we apply Remark 2.2.6.15 and Example 2.2.6.17 in [Lur17] to obtain an explicit formula for the Day convolution of two functors. In fact, this precisely mirrors the classical formula for Day convolution in 1-categories. Letting F, F' be functors $\mathcal{C}\to\mathcal{D}$, we have the formula

$$(F\otimes_{\mathcal{F}\mathrm{un}(\mathcal{C},\mathcal{D})} F')(C) \simeq \operatorname*{colim}_{C_0\otimes_{\mathcal{C}} C_1\to C} F(C_0)\otimes_{\mathcal{D}} F'(C_1).$$

4.2 Presentable ∞-categories

Most ∞-categories of interest to us are of a type called *presentable* ∞-categories, which are a convenient setting for higher category theory. We will recall some of the basic theory of presentable ∞-categories, as in Chapter 5 of [Lur09] as the primary reference.

There are many different ways one can think about presentable ∞-categories. In this section, we mostly illustrate the idea that presentable ∞-categories are generated under filtered colimits by some smaller, more tractable subcategory.

The end result of this section is to construct an ∞-category of presentable stable ∞-categories, and recall its symmetric monoidal structure.

4.2.1 Ind-completion and presheaves

To reify the picture of presentable ∞-categories sketched above, we will first quantify the notion of a filtered colimit completion, which is given by Ind-*completion*. Recall that for any ∞-category $\mathcal{C}$, we defined the ∞-category of presheaves $\mathcal{P}(\mathcal{C})$ as the functor ∞-category

$$\mathcal{P}(\mathcal{C}) := \mathcal{F}\mathrm{un}(\mathcal{C}^{\mathrm{op}},\mathcal{S}),$$

which comes equipped with a *Yoneda embedding*

$$よ : \mathcal{C}\to\mathcal{P}(\mathcal{C}).$$

This ∞-category of presheaves can be regarded as the free colimit completion of $\mathcal{C}$. Indeed ([Lur09], Theorem 5.1.5.6) states that for any small ∞-category $\mathcal{C}$ and ∞-category $\mathcal{D}$ with small colimits, precomposition with the Yoneda embedding induces an equivalence of ∞-categories

$$\mathcal{F}\mathrm{un}^{\mathrm{L}}(\mathcal{P}(\mathcal{C}),\mathcal{D}) \xrightarrow{\sim} \mathcal{F}\mathrm{un}(\mathcal{C},\mathcal{D}), \tag{1}$$

where the source is the ∞-category of functors that preserve small colimits. Our goal in this section is to obtain a similar construction, where *colimits* are now replaced by *filtered colimits*.

Recall that any functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ can be *unstraightened* to obtain a fibration $\int f \to \mathcal{C}$. If we think of this total ∞-category $\int f$ as the diagram of which $f \in \mathcal{P}(\mathcal{C})$ is the formal colimit, then restricting to presheaves whose associated total ∞-category is filtered should recover the appropriate notion of Ind-completion. To avoid set-theoretic difficulties, we will introduce a smallness condition, given by the notion of κ-filtered ∞-category for some regular cardinal κ.

Definition 4.1 ([Lur09] 5.3.1.7)**.** an ∞-category $\mathcal{D}$ is called **κ-filtered** if for any κ-small simplicial set S with a map

$$p : S \to \mathcal{D},$$

there exists an extension of p to the right cone of S.

Note that when $\kappa = \omega$, we refer to ω-filtered ∞-categories simply as filtered ∞-categories etc. We are now ready to define the Ind-completion.

Definition 4.2 ([Lur09] 5.3.5.1)**.** Given a small ∞-category $\mathcal{C}$ and a regular cardinal κ, we define $\mathrm{Ind}_\kappa(\mathcal{C})$ to be the full subcategory of $\mathcal{P}(\mathcal{C})$ on the functors $f : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ such that the total ∞-category $\int f \to \mathcal{C}$ in the corresponding unstraightening is κ-filtered.

In particular, one can consider the objects of $\mathcal{P}(\mathcal{C})$ in the image of the Yoneda embedding, i.e. of the form $\text{よ}(C)$ for $C \in \mathcal{C}$. Since these unstraighten to the slice fibrations $\mathcal{C}_{/C} \to \mathcal{C}$, and the latter total ∞-category is filtered for any κ since it has a terminal object, we conclude that the Yoneda embedding factors through the inclusion $\mathrm{Ind}_\kappa(\mathcal{C}) \to \mathcal{P}(\mathcal{C})$. In fact, by ([Lur09] Proposition 5.3.5.3) this subcategory has all κ-filtered colimits.

We are interested in the way $\mathcal{C}$ is contained in $\mathrm{Ind}_\kappa(\mathcal{C})$ using the Yoneda embedding. It turns out that every object in the image of the Yoneda embedding is in fact compact in the Ind-completion. Let us first recall the notion of compactness

Definition 4.3 ([Lur09] 5.3.4.5)**.** Let $\mathcal{D}$ be an ∞-category admitting κ-filtered colimits (i.e. colimits of diagrams out of κ-filtered ∞-categories). Then an object D of $\mathcal{D}$ is said to be **κ-compact** if the corresponding functor

$$\mathrm{map}_{\mathcal{D}}(D, -) : \mathcal{D}^{\mathrm{op}} \to \mathcal{S}$$

preserves κ-filtered colimits.

It is then rather immediate that any object of the form $\text{よ}(C)$ in $\mathrm{Ind}_\kappa(\mathcal{C})$ is κ-compact. Indeed, the functor

$$\mathrm{map}_{\mathrm{Ind}_\kappa(\mathcal{C})}(\text{よ}(C), -) : \mathrm{Ind}_\kappa(\mathcal{C}) \to \mathcal{S}$$

is nothing but evaluation at the object C by the Yoneda Lemma. Since filtered colimits in $\mathrm{Ind}_\kappa(\mathcal{C})$ can be computed in the ambient ∞-category $\mathcal{P}(\mathcal{C})$, where they are computed levelwise, we conclude that this functor commutes with κ-filtered colimits as required.

We conclude with a concrete universal property of Ind-completion analogous to that of the presheaf ∞-category

Proposition 4.1 ([Lur09] 5.3.5.10)**.** *Let $\mathcal{C}$ be a small ∞-category, and $\mathcal{D}$ an ∞-category with κ-filtered colimits. Then composition with the Yoneda embedding induces an equivalence of ∞-categories*

$$\mathcal{F}\mathrm{un}^{\kappa\text{-cont}}(\mathrm{Ind}_\kappa(\mathcal{C}), \mathcal{D}) \xrightarrow{\sim} \mathcal{F}\mathrm{un}(\mathcal{C}, \mathcal{D}),$$

where the source is the full subcategory on functors preserving κ-filtered colimits.

4.2.2 Accessibility and presentability

Having developed the theory of Ind-completions, i.e. filtered colimits completions of ∞-categories, we can now quantify what it means for an ∞-category to be generated under filtered colimits by a set of (compact) generators. This is the notion of an accessible ∞-category

Definition 4.4 ([Lur09] 5.4.2.1). An ∞-category $\mathcal{C}$ is said to be **κ-accessible** for some regular cardinal κ if there exists a small ∞-category $\mathcal{C}'$ and an equivalence

$$\mathrm{Ind}_\kappa(\mathcal{C}') \xrightarrow{\sim} \mathcal{C}.$$

A functor is said to be κ-accessible if it preserves κ-filtered colimits.

If an ∞-category or functor is κ-accessible for some κ, we will just say that it is accessible. It is then verified in [Lur09] Proposition 5.4.2.2 that this corresponds with our intuition about κ-compact generation under small κ-filtered colimits.

An important class of accessible ∞-categories arises as presheaves on a small ∞-category. This is the content of [Lur09] Proposition 5.2.5.12, which states that for any small ∞-category $\mathcal{C}$, the subcategory of κ-compact objects in $\mathcal{P}(\mathcal{C})$ is equivalent to a small ∞-category $\mathcal{D}$, and is such that $\mathcal{P}(\mathcal{C})$ can be identified with $\mathrm{Ind}_\kappa(\mathcal{D})$. If we set $\mathcal{C}$ to be Δ^0, such that $\mathcal{P}(\mathcal{C}) \simeq \mathcal{S}$, we conclude that the ∞-category of spaces is accessible.

Having defined accessible ∞-categories, we can now come to the main definition of this section, namely that of a presentable ∞-category.

Definition 4.5 ([Lur09] 5.5.0.1). An accessible ∞-category with all small colimits is called **presentable**.

Since colimits in functor ∞-categories such as presheaf ∞-categories are computed levelwise, and the ∞-category of spaces $\mathcal{S}$ has all small colimits, we obtain an important class of presentable ∞-categories: namely any presheaf ∞-category $\mathcal{P}(\mathcal{C})$ where $\mathcal{C}$ is a small ∞-category. In fact, by an observation of Simpson, all presentable ∞-categories arise as accessible localisations of such presheaf ∞-categories, cf. [Lur09] Theorem 5.5.1.1.

Presentable ∞-categories enjoy a variety of nice properties, such as the adjoint functor theorem. Recall that in $(1,1)$-category theory, a functor admits a right adjoint if and only if it preserves small colimits. This result does not generalise to arbitrary ∞-categories, but it does hold for functors between presentable ∞-categories

Theorem 4.1 (Adjoint Functor Theorem, [Lur09] 5.5.2.9). *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between presentable ∞-categories, then F has a right adjoint if and only if it preserves small colimits, and dually F has a left adjoint if and only if it is accessible and preserves small limits.*

This theorem will often be used explicitly to posit the existence of right or left adjoints to functors pereserving small colimits resp. limits. This result allows us to construct ∞-categories of presentable ∞-categories, using left or right adjoint functors.

Definition 4.6 ([Lur09] 5.5.3.1). Given the ∞-category $\widehat{\mathcal{C}\mathrm{at}}_\infty$ of not-necessarily-small ∞-categories, let us consider the subcategories $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ and $\mathcal{P}\mathrm{r}^{\mathrm{R}}$ whose objects are presentable ∞-categories and whose morphisms are functors that preserve small colimits resp. small limits, and are accessible in the latter case.

By the Adjoint Functor Theorem stated above, it is clear that these two ∞-categories are anti-equivalent, since the datum of a small-colimit preserving functor is the same as the datum of the left adjoint functor when restricted to presentable ∞-categories.

The goal of the remainder of this section is to equip $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ with some nice categorical structures, such as internal mapping objects and a symmetric monoidal structure.

An immediate candidate for the internal mapping object between objects $\mathcal{C}$ and $\mathcal{D}$ in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ is the full subcategory $\mathcal{F}\mathrm{un}^{\mathrm{L}}(\mathcal{C},\mathcal{D}) \subset \mathcal{F}\mathrm{un}(\mathcal{C},\mathcal{D})$ on small-colimit-preserving (i.e. left adjoint) functors. It turns out that this is the correct notion, cf. [Lur09] Proposition 5.5.3.8, where it is shown that this subcategory is presentable, hence can be considered as an object of $\mathcal{P}\mathrm{r}^{\mathrm{L}}$.

Having constructed internal mapping objects, we now construct the symmetric monoidal structure on $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ such that it becomes a closed symmetric monoidal ∞-category. Consider a morphism $\mathcal{C} \to \mathcal{F}\mathrm{un}^{\mathrm{L}}(\mathcal{D}, \mathcal{E})$ in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$. Viewing this as a morphism in the ambient ∞-category $\widehat{\mathcal{C}\mathrm{at}}_\infty$, we can use the Cartesian closed symmetric monoidal structure in the latter to see that the datum of such a morphism is equivalent to the datum of a functor $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ which preserves colimits separately in each variable. If we tentatively let $\otimes$ denote the expected symmetric monoidal structure on $\mathcal{P}\mathrm{r}^{\mathrm{L}}$, then $\mathcal{C} \otimes \mathcal{D}$ should have the universal property that a colimit-preserving functor out of it is equivalent to the datum of a functor out of $\mathcal{C} \times \mathcal{D}$ that preserves colimits in each variable separately. We will now make this rigorous.

Proposition 4.2. *The ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ defined above has a symmetric monoidal structure, denoted $\otimes$, with the universal property that small-colimit-preserving functors out of $\mathcal{C} \otimes \mathcal{D}$ are equivalent to functors out of $\mathcal{C} \times \mathcal{D}$ preserving colimits in each variable separately.*

The main reference for this construction is [Lur07], and we present a slight reformulation of the proof of this statement in op. cit.

Proof. Recall that a symmetric monoidal ∞-category is an ∞-operad such that its structure morphism down to the category of finite pointed sets $\mathbb{F}_*$ is a coCartesian fibration. Now we know that $\widehat{\mathcal{C}\mathrm{at}}_\infty$ has a symmetric monoidal structure given by the Cartesian product, so that its ∞-category of operators

$$p : \widehat{\mathcal{C}\mathrm{at}}_\infty^\times \to \mathbb{F}_*$$

forms a symmetric monoidal ∞-category. Now consider the subcategory $(\mathcal{P}\mathrm{r}^{\mathrm{L}})^\otimes$ of $\widehat{\mathcal{C}\mathrm{at}}_\infty^\times$ such that

- Objects of $(\mathcal{P}\mathrm{r}^{\mathrm{L}})^\otimes$ are tuples $(\{\mathcal{C}_i\}_{i \in \langle n \rangle^\circ}, \langle n \rangle)$ such that every $\mathcal{C}_i$ is presentable, i.e. an object of $\mathcal{P}\mathrm{r}^{\mathrm{L}}$.
- A morphism
$$(\{\mathcal{C}_i\}_{i \in \langle n \rangle^\circ}, \langle n \rangle) \to (\{\mathcal{D}_j\}_{j \in \langle m \rangle^\circ}, \langle m \rangle)$$
over $\alpha : \langle n \rangle \to \langle m \rangle$ lies in $(\mathcal{P}\mathrm{r}^{\mathrm{L}})^\otimes$ if for every j in $\langle m \rangle^\circ$, the corresponding morphism
$$\prod_{i \in \alpha^{-1}(j)} \mathcal{C}_i \to \mathcal{D}_j$$
preserves colimits in each variable separately.

It is clear that $(\mathcal{P}\mathrm{r}^{\mathrm{L}})^\otimes_{\langle 1 \rangle} \simeq \mathcal{P}\mathrm{r}^{\mathrm{L}}$, so that this is a candidate for a symmetric monoidal structure on $\mathcal{P}\mathrm{r}^{\mathrm{L}}$. To prove that the subcategory constructed above is part of the datum of a symmetric monoidal ∞-category, we need to show that the restriction
$$q : (\mathcal{P}\mathrm{r}^{\mathrm{L}})^\otimes \to \widehat{\mathcal{C}\mathrm{at}}_\infty^\times \to \mathbb{F}_*$$
is still a coCartesian fibration. We begin by showing that an edge $\langle n \rangle \to \langle m \rangle$ in $\mathbb{F}_*$ can be lifted to a locally q-coCartesian edge in $(\mathcal{P}\mathrm{r}^{\mathrm{L}})^\otimes$ with fixed source. Note that the target of such a morphism would land in the fibre
$$(\mathcal{P}\mathrm{r}^{\mathrm{L}})^\otimes_{\langle m \rangle} \subset (\widehat{\mathcal{C}\mathrm{at}}_\infty^\times)_{\langle m \rangle} \simeq (\widehat{\mathcal{C}\mathrm{at}}_\infty)^m,$$
so that we may reduce to the case where $\langle m \rangle = 1$. This allows us to rewrite this question in a more obvious way.

Given some lift of the source, i.e. an object of $(\mathcal{P}\mathrm{r}^{\mathrm{L}})^\otimes_{\langle n \rangle}$, which we can identify with a Cartesian product $\mathcal{C}_1 \times \cdots \times \mathcal{C}_n$ of presentable ∞-categories, can we find a functor

$$F : \mathcal{C}_1 \times \cdots \times \mathcal{C}_n \to \mathcal{D}$$

into a presentable ∞-category $\mathcal{D}$ which preserves small colimits in each variable separately, and which is locally q-coCartesian, i.e. is such that for all presentable $\mathcal{E}$, precomposition with F induces a homotopy equivalence between $\iota\mathcal{F}\mathrm{un}^{\mathrm{L}}(\mathcal{D},\mathcal{E})$ and $\iota\mathcal{F}\mathrm{un}^{\mathrm{L}^n}(\mathcal{C}_1\times\cdots\times\mathcal{C}_n,\mathcal{E})$, where the latter is the (core of the) ∞-category of functors that preserve colimits in each variable separately[1].

The answer to this question is positive, and such an object can be constructed explicitly by induction. Since this object will be the tensor product of $\mathcal{C}_1,\cdots\mathcal{C}_n$ in the symmetric monoidal structure on $\mathcal{P}\mathrm{r}^{\mathrm{L}}$, we pre-emptively denote it by $\mathcal{C}_1\otimes\cdots\otimes\mathcal{C}_n$ It is clear that when $n=0$ we can take $\mathcal{D}=\mathcal{S}=\mathcal{P}(\Delta^0)$ to be the ∞-category of spaces, so that the Yoneda embedding in equation 1 gives us the desired equivalence

$$\mathcal{F}\mathrm{un}^{\mathrm{L}}(\mathcal{S},\mathcal{E})\simeq\mathcal{F}\mathrm{un}^{\mathrm{L}}(\mathcal{P}(\Delta^0),\mathcal{E})\simeq\mathcal{F}\mathrm{un}(\Delta^0,\mathcal{E}).$$

If $n=1$, there is nothing to prove, so we assume that $n>1$. In that case we can apply the adjunction

$$\mathcal{F}\mathrm{un}^{\mathrm{L}^n}(\prod_{i=1}^{n}\mathcal{C}_i,\mathcal{E})\simeq\mathcal{F}\mathrm{un}^{\mathrm{L}^{n-1}}(\prod_{i=1}^{n-1}\mathcal{C}_i,\mathcal{F}\mathrm{un}^{\mathrm{L}}(\mathcal{C}_n,\mathcal{E}))$$

coming from the Cartesian closed monoidal structure on $\widehat{\mathcal{C}\mathrm{at}}_\infty$. Now note that the target $\mathcal{F}\mathrm{un}^{\mathrm{L}}(\mathcal{C}_n,\mathcal{E})$ is once again presentable, so that we can apply the induction hypothesis to see that precomposition with the canonical morphism induces an equivalence

$$\mathcal{F}\mathrm{un}^{\mathrm{L}^n}(\prod_{i=1}^{n}\mathcal{C}_i,\mathcal{E})\simeq\mathcal{F}\mathrm{un}^{\mathrm{L}}(\bigotimes_{i=1}^{n-1}\mathcal{C}_i,\mathcal{F}\mathrm{un}^{\mathrm{L}}(\mathcal{C}_n,\mathcal{E})).$$

When $n=2$ we can work explicitly. We use the anti-equivalence between $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ and $\mathcal{P}\mathrm{r}^{\mathrm{R}}$ to identify the mapping object $\mathcal{F}\mathrm{un}^{\mathrm{L}}(\mathcal{C}_2,\mathcal{E})$ in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ with the full subcategory on accessible functors in the mapping object $\mathcal{F}\mathrm{un}^{\mathrm{R}}(\mathcal{E},\mathcal{C}_2)^{\mathrm{op}}$. The notation $\mathcal{F}\mathrm{un}^{\mathrm{R}}$ denotes the full subcategory on functors that preserve all small limits. We then obtain a chain of equivalences and an inclusion

$$\begin{aligned}
\mathcal{F}\mathrm{un}^{\mathrm{L}^2}(\mathcal{C}_1\times\mathcal{C}_2,\mathcal{E}) &\simeq\mathcal{F}\mathrm{un}^{\mathrm{L}}(\mathcal{C}_1,\mathcal{F}\mathrm{un}^{\mathrm{L}}(\mathcal{C}_2,\mathcal{E})),\\
&\simeq\mathcal{F}\mathrm{un}^{\mathrm{L}}(\mathcal{C}_1,\mathcal{F}\mathrm{un}^{\mathrm{R}}(\mathcal{E},\mathcal{C}_2)^{\mathrm{op}}),\\
&\to\mathcal{F}\mathrm{un}^{\mathrm{L}}(\mathcal{C}_1,\mathcal{F}\mathrm{un}^{\mathrm{L}}(\mathcal{E}^{\mathrm{op}},\mathcal{C}_2^{\mathrm{op}})),\\
&\simeq\mathcal{F}\mathrm{un}^{\mathrm{L}^2}(\mathcal{C}_1\times\mathcal{E}^{\mathrm{op}},\mathcal{C}_2^{\mathrm{op}}),\\
&\simeq\mathcal{F}\mathrm{un}^{\mathrm{L}}(\mathcal{E}^{\mathrm{op}},\mathcal{F}\mathrm{un}^{\mathrm{L}}(\mathcal{C}_1,\mathcal{C}_2^{\mathrm{op}}))),\\
&\simeq\mathcal{F}\mathrm{un}^{\mathrm{L}}(\mathcal{E}^{\mathrm{op}},\mathcal{F}\mathrm{un}^{\mathrm{R}}(\mathcal{C}_2^{\mathrm{op}},\mathcal{C}_1)^{\mathrm{op}}),\\
&\simeq\mathcal{F}\mathrm{un}^{\mathrm{R}}(\mathcal{E},\mathcal{F}\mathrm{un}^{\mathrm{R}}(\mathcal{C}_2^{\mathrm{op}},\mathcal{C}_1))^{\mathrm{op}},\\
&\simeq\mathcal{F}\mathrm{un}^{\mathrm{L}}(\mathcal{F}\mathrm{un}^{\mathrm{R}}(\mathcal{C}_2^{\mathrm{op}},\mathcal{C}_1),\mathcal{E}),
\end{aligned}$$

that identifies the source with the full subcategory on accessible functors in $\mathcal{F}\mathrm{un}^{\mathrm{R}}(\mathcal{E},\mathcal{F}\mathrm{un}^{\mathrm{R}}(\mathcal{C}_2^{\mathrm{op}},\mathcal{C}_2))^{\mathrm{op}}$. We therefore define $\mathcal{C}_1\otimes\mathcal{C}_2:=\mathcal{F}\mathrm{un}^{\mathrm{R}}(\mathcal{C}_2^{\mathrm{op}},\mathcal{C}_1)$. The fact that the latter is once again a presentable ∞-category is the content of [Lur07] Lemma 4.1.4, and essentially uses the remark above that any presentable ∞-category arises as a left exact localisation of an ∞-category of presheaves on a small ∞-category. Finally, for $n>2$, we identify $(\mathcal{C}_1\otimes\cdots\otimes\mathcal{C}_{n-1})$ with $(\mathcal{C}_1\otimes\cdots\otimes\mathcal{C}_{n-1})\otimes\mathcal{C}_{n-1}$ and apply the induction hypothesis.

We conclude that $q:(\mathcal{P}\mathrm{r}^{\mathrm{L}})^{\otimes}\to\mathbb{F}_*$ is a symmetric monoidal ∞-category whose underlying ∞-category $(\mathcal{P}\mathrm{r}^{\mathrm{L}})^{\otimes}_{\langle 1\rangle}$ is $\mathcal{P}\mathrm{r}^{\mathrm{L}}$, and such that the operation $\otimes$ can be characterised by the desired universal property. □

Remark 4.1. The ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ satisfies a number of closure properties that we will not prove here. In particular, it is closed under localisations, i.e. a localisation of a presentable ∞-category is once again presentable. Further, it is cotensored over $\mathcal{C}\mathrm{at}_\infty$ in the obvious way, so that if $\mathcal{C}$ is presentable and K is any

[1]Since we are working in the $(\infty,1)$-category of (presentable) ∞-categories, we must take the core of mapping ∞-categories. However, the proof below actually extends to an equivalence on mapping ∞-categories.

∞-category, the functor ∞-category $\mathcal{F}\mathrm{un}(K, \mathcal{C})$ is once again presentable. In the next section 4.3, we will see that the full subcategories $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{*}$, $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}$ on pointed and stable objects satisfy the same closure properties. In fact, we will often use these facts implicitly, assuming it is obvious that many functor ∞-categories into (pointed/stable) presentable ∞-categories and their localisations are once again (pointed/stable) presentable.

4.3 Spectra and stabilisation

In this section, we will recall the definition of stable ∞-categories, and how they can be constructed in a universal way using stabilisations. We will argue why the universal stable ∞-category is the ∞-category of spectra Sp. The main references for this section are [Lur17] and [GGN15].

Definition 4.7. An ∞-category $\mathcal{D}$ is said to be pointed if it has a zero object, i.e. an object which is both initial and final in $\mathcal{D}$

The presence of a zero object in an ∞-category $\mathcal{D}$ allows us to consider the notion of fibre and cofibre sequences in $\mathcal{D}$. Indeed, if $f : Y \to Z$ resp. $g : X \to Y$ is a morphism in $\mathcal{D}$, and 0 is a zero object of $\mathcal{D}$, we define its fibre $\mathrm{fib}(f)$ resp. cofibre $\mathrm{cof}(g)$ to the pullback resp. pushout

$$\begin{tikzcd} \mathrm{fib}(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & Y \arrow[d, "f"] \\ 0 \arrow[r] & Z \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "g"] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] \arrow[ur, phantom, "\ulcorner" very near end] & \mathrm{cof}(f). \end{tikzcd}$$

Being constructed as limits and colimits along a diagram containing a zero object (which is unique up to contractible choice as a limit/colimit itself), we infer that fibres and cofibres–if they exist–are unique up to equivalence. One can then consider a special class of pointed ∞-categories, namely those where fibres and cofibres all exist and agree. We come to the following definition:

Definition 4.8 ([Lur17], 1.1.1.9)**.** A pointed ∞-category is said to be **stable** if all morphisms have fibres and cofibres and these agree, in the sense that $X \to Y \to Z$ is the upper right corner of a fibre diagram if and only if it is the upper right corner of a cofibre diagram.

in fact, for a pointed ∞-category $\mathcal{D}$, the requirement that $\mathcal{D}$ be stable is equivalent to the requirement that it admits all finite limits and colimits, and that pushouts and pullbacks agree. This is the content of [Lur17] Proposition 1.1.3.4.

Since we are primarily interested in presentable ∞-categories, we immediately restrict to the presentable case. We can then construct two full subcategories of $\mathcal{P}\mathrm{r}^{\mathrm{L}}$, denoted $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{*}$ and $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}$ respectively, on the pointed resp. stable presentable ∞-categories. The goal of this section is to show that the inclusions of these full subcategories are nice enough that they admit localisation functors going the other way, which one can view as the corresponding universal constructions.

In fact, these localisations are of a special type called *smashing localisations* in [GGN15], whose definition is recalled here.

Definition 4.9 ([GGN15] 3.2)**.** A localisation $L : \mathcal{D} \to \mathcal{D}$ of a symmetric monoidal ∞-category $\mathcal{D}$ is called a **smashing** localisation if it is given by tensoring $L(-) = - \otimes I$ for some object I of $\mathcal{D}$.

The condition that L be a localisation will force the object I to be an idempotent commutative algebra object of $\mathcal{D}$, and in fact establishes a correspondence between smashing localisations and such objects. As shown in Section 3 of op. cit, these smashing localisations (where we now restrict to our objects of interest, i.e. smashing localisations of the closed symmetric monoidal ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$) enjoy a multitude of nice properties, summarised in the following proposition.

Proposition 4.3 ([GGN15] 3.9)**.** *Let $L : \mathcal{P}\mathrm{r}^{\mathrm{L}} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$ be a smashing localisation. Let $\mathcal{C}$ and $\mathcal{D}$ be objects of $\mathcal{P}\mathrm{r}^{\mathrm{L}}$. Then*

- *The natural map $\mathcal{C} \to L\mathcal{C}$ induces an equivalence*

$$\mathcal{F}\mathrm{un}^{\mathrm{L}}(L\mathcal{C}, \mathcal{D}) \xrightarrow{\sim} \mathcal{F}\mathrm{un}^{\mathrm{L}}(\mathcal{C}, \mathcal{D})$$

whenever $\mathcal{D}$ lies in the essential image of L. If both $\mathcal{C}$ and $\mathcal{D}$ are closed symmetric monoidal[2]*, then this equivalence refines to an equivalence on ∞-categories of symmetric monoidal functors.*

- *If $\mathcal{C}$ is once again assumed to be closed symmetric monoidal, then $L\mathcal{C}$ inherits a unique closed symmetric monoidal structure such that $\mathcal{C} \to L\mathcal{C}$ is symmetric monoidal.*
- *The image $L\mathcal{P}\mathrm{r}^{\mathrm{L}}$ is equivalent to* $\mathrm{Mod}(\mathcal{P}\mathrm{r}^{\mathrm{L}}; L\mathcal{S})$.

Remark 4.2. Note that the first point of this proposition is a little stronger than simply a consequence of the adjunction between tensoring with $L\mathcal{S}$ and forgetting the $L\mathcal{S}$-module structure. The latter would induce an equivalence on mapping spaces of left adjoint functors in the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$, while the first statement actually gives an equivalence on functor categories, i.e. on the cotensors in the self-enriched ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$. The two are related to each other by taking the core of the cotensor to obtain the mapping space.

To apply this proposition, we will show that the formation of the ∞-category of pointed objects in a presentable ∞-category is a smashing localisation. For the remainder of this section, $\mathcal{C}$ will denote a presentable ∞-category.

On the level of objects, we have good evidence towards this claim. Indeed, if we consider the tensor product $\mathcal{C} \otimes \mathcal{S}_*$, it is clear that this is an object of $\mathcal{P}\mathrm{r}^{\mathrm{L}}_*$. We therefore have a functor

$$- \otimes \mathcal{S}_* : \mathcal{P}\mathrm{r}^{\mathrm{L}} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}_*$$

which is our candidate for a smashing localisation. If it is a localisation, it is clear that it is smashing, being defined as such. The former statement is equivalent to showing that $\mathcal{S}_*$ is an idempotent commutative algebra object of $\mathcal{P}\mathrm{r}^{\mathrm{L}}$, as remarked in [GGN15]. This is the content of [Lur17] Proposition 4.8.2.11, in which $\mathcal{S}_*$ is shown to have a closed symmetric monoidal structure with unit $S^0 = * \sqcup *$, uniquely characterised by the property that the unit map $\mathcal{S} \to \mathcal{S}_*$ (explicitly given by adjoining a disjoint basepoint to a space) is symmetric monoidal.

We conclude that $\mathcal{P}\mathrm{r}^{\mathrm{L}}_* \subset \mathcal{P}\mathrm{r}^{\mathrm{L}}$ is a localising subcategory with corresponding smashing localisation given by tensoring with the idempotent object $\mathcal{S}_*$. Application of the proposition above then tells us that if $\mathcal{D}$ is a pointed presentable ∞-category, whence there is an equivalence

$$\mathcal{F}\mathrm{un}^{\mathrm{L}}(\mathcal{C} \otimes \mathcal{S}_*, \mathcal{D}) \xrightarrow{\sim} \mathcal{F}\mathrm{un}^{\mathrm{L}}(\mathcal{C}, \mathcal{D}).$$

The ∞-category of pointed objects $\mathcal{C} \otimes \mathcal{S}_*$ will henceforth be denoted by $\mathcal{C}_*$.

The stable case is analogous, yet is less intuitive than the pointed case. We therefore reformulate the stability condition for presentable ∞-categories. In any pointed ∞-category $\mathcal{C}$, which will be assumed to be presentable from now on, given an object X one can form pullback or pushout diagrams

$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & 0 \arrow[d] \\ 0 \arrow[r] & X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] \arrow[ur, phantom, very near end, "\ulcorner"] & \Sigma X. \end{tikzcd}$$

These constructions can be promoted to adjoint endofunctors $\Sigma \dashv \Omega$ of $\mathcal{C}$ following [Lur17] Remark 1.1.2.8, called the suspension and loop functors respectively. We can then apply [Lur17] Proposition 1.4.2.11, which tells us that a pointed presentable ∞-category is stable if and only if the loop functor associated to it is an equivalence.

Remark 4.3. In fact, note that the construction of the suspension functor gives us a sort of triangulated structure on a stable ∞-category, which will actually induce the structure of a triangulated category on its homotopy category. Indeed, consider a pair of (co)fibre sequences

$$X \to Y \to Z, Y \to Z \to W.$$

[2]A commutative algebra object in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ corresponds to a symmetric monoidal ∞-category under the forgetful functor from $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ to $\mathcal{C}\mathrm{at}_\infty$, but the additional requirement that the tensor product preserves all small colimits in each variable guarantees that the resulting ∞-cat is closed symmetric monoidal, which we will also refer to as a presentably symmetric monoidal ∞-category.

These can be encoded in a pair of pushout diagrams

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & Z \arrow[r] \arrow[ul, phantom, "\ulcorner"] & W. \arrow[ul, phantom, "\ulcorner"]
\end{tikzcd}$$

But now one sees that the total square is a pasting of coCartesian squares ergo coCartesian itself, so that $W \simeq \Sigma X$, and we see that the suspension should be thought of as inducing the shift on the triangulated homotopy category of a stable ∞-category. In fact, we will frequently use the converse, which states that if $W \simeq \Sigma X$, whence the total square is biCartesian, then the left square being biCartesian implies that the right square is biCartesian. This means that

$$X \to Y \to Z$$

is a fibre sequence if and only if

$$Y \to Z \to \Sigma X$$

is a fibre sequence.

The functor taking a pointed presentable ∞-category to its stabilisation, i.e. the left adjoint to the inclusion $\mathcal{P}r^{\mathrm{L}}_{*} \subset \mathcal{P}r^{\mathrm{L}}_{\mathrm{St}}$ admits an explicit description that also elucidates its relation with more classical or intuitive notions of spectrum objects. For this, we will need to consider the ∞-category of finite spaces denoted $\mathcal{S}^{\mathrm{fin}}$. This is defined as the ∞-category containing a terminal object $*$ and closed under finite colimits. In fact, we will make use of its pointed version, denoted $\mathcal{S}^{\mathrm{fin}}_{*}$, which by the above discussion can be obtained as

$$\mathcal{S}^{\mathrm{fin}}_{*} \simeq \mathcal{S}^{\mathrm{fin}} \otimes \mathcal{S}_{*}.$$

The objects of this ∞-category admit the intuitive description of pointed finite spaces. In particular, it contains the spheres $S^n, n \geq 0$, since these are obtained from the terminal object $*$ using finite colimits (adjoining a disjoint basepoint to obtain S^0, and then using suspensions to obtain S^n for $n \geq 1$). This allows us to define the ∞-category of spectrum objects.

Definition 4.10. Let $\mathcal{C}$ be an ∞-category with finite limits, which will usually be an object of $\mathcal{P}r^{\mathrm{L}}_{*}$. Then one defines the ∞-category of **spectrum objects** in $\mathcal{C}$ or the **stabilisation** of $\mathcal{C}$ as

$$\mathrm{Sp}(\mathcal{C}) := \mathrm{Exc}_{*}(\mathcal{S}^{\mathrm{fin}}_{*}, \mathcal{C}).$$

This is the full subcategory of $\mathcal{F}\mathrm{un}(\mathcal{S}^{\mathrm{fin}}_{*}, \mathcal{C})$ on functors which

1. are reduced, i.e. carry the final object $*$ of $\mathcal{S}^{\mathrm{fin}}_{*}$ to a final object in $\mathcal{C}$,
2. and are excisive, so that they send pushout squares in the source to pullback squares in the target.

This apparently involved construction can be seen to recover the classical notion of Ω-spectra using the following heuristic. Since there is a pushout square of the form

$$\begin{tikzcd}
S^{n-1} \arrow[r] \arrow[d] & * \arrow[d] \\
* \arrow[r] & S^n \arrow[ul, phantom, "\ulcorner"]
\end{tikzcd}$$

witnessing that S^n is the suspension of S^{n-1} for every $n \geq 1$, then an object F of $\mathrm{Sp}(\mathcal{C})$ sends this puhsout diagram to a pullback diagram

$$\begin{tikzcd}
F(S^{n-1}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & * \arrow[d] \\
* \arrow[r] & F(S^n).
\end{tikzcd}$$

Using the notation established above, this means that $F(S^{n-1}) \simeq \Omega F(S^n)$. If $\mathcal{C}$ is the category of pointed spaces, it is then clear that an object of $\mathrm{Sp}(\mathcal{S}_*)$ can be seen as an ∞-categorical version of an Ω-spectrum.

It is then standard to verify that the resulting ∞-category is still presentable, cf. [Lur17] Remark 1.4.2.4 since it is an accessible localisation of its ambient functor ∞-category. Furthermore, it is pointed and admits finite limits.

The next step is to verify that the resulting ∞-category is actually an element of $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}$, for which one simply needs to verify that it is stable. As asserted above, it is sufficient to verify that the loop or suspension functors of the pointed presentable ∞-category $\mathrm{Sp}(\mathcal{C})$ are equivalences. This is now immediate, since it is clear the the endofunctor of $\mathrm{Exc}_*(\mathcal{S}^{\mathrm{fin}}_*, \mathcal{C})$ given by precomposition with the suspension Σ of finite pointed spaces is inverse to the loop functor Ω intrinsic to this pointed ∞-category with finite limits. Indeed, limits commute with Cartesian squares and preserve zero objects, so that this loop functor is compute pointwise, and it is then immediate from the previous paragraph that this is inverse to precomposition with Σ.

Remark 4.4. A particular upshot of the discussion above is that it endows stabilisation with a very concrete universal property. Indeed, since stabilisation is a smashing localisation of $\mathcal{P}\mathrm{r}^{\mathrm{L}}$, we see that there is an equivalence

$$\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}} \simeq \mathrm{Mod}(\mathcal{P}\mathrm{r}^{\mathrm{L}}; \mathrm{Sp}(\mathcal{S})),$$

where the algebra object we consider is the stabilisation of the presentable ∞-category of spaces. The latter is none other than Sp[3]. In fact, from this description we also see that stabilisation, i.e. tensoring with Sp is left adjoint (in an enriched way) to the obvious forgetful functor

$$\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}} \to \mathcal{P}\mathrm{r}^{\mathrm{L}},$$

which can now be viewed as forgetting the Sp-module structure. This means that if $\mathcal{D}$ is an element of $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}$ and $\mathcal{C}$ an element of $\mathcal{P}\mathrm{r}^{\mathrm{L}}$, there is an equivalence

$$\mathcal{F}\mathrm{un}^{\mathrm{L}}(\mathcal{C}, \mathcal{D}) \simeq \mathcal{F}\mathrm{un}^{\mathrm{L}}(\mathrm{Sp}(\mathcal{C}), \mathcal{D}).$$

Remark 4.5. The stabilisation of a pointed ∞-category with finite limits can be done in greater generality, but the upshot of working with presentable ∞-categories right away is that one does not need to worry about exactness. Indeed, in the more general setting, a functor between stable ∞-categories ought to preserve (co)fibre sequences, to respect the suspension and loop functors, in analogy with an exact functor of abelian categories. In the case of presentable ∞-categories, we already restrict to left adjoints preserving all colimits such as cofibre sequences, and there is no more restriction needed.

4.4 Monadic adjunctions of rigidly generated ∞-categories

Many modern advances in homotopy theory and higher category theory appear to follow a pattern where one can consider presentable stable ∞-caetgories as modules over stable presentably symmetric monoidal ∞-categories. Indeed, since $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}$ is symmetric monoidal, and stable presentably symmetric monoidal ∞-categories correspond precisely to commutative algebra objects in this ∞-category, we see that this notion can be made rigorous. In particular, by the above description of stabilisation as a smashing localisation, we ought to think of Sp as the unit in $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}$, and think of the theory of presentable stable ∞-categories as Sp-linear category theory. This hints at some analogy towards the theory of abelian categories, each of them being enriched in the universal abelian category $\mathrm{Mod}_{\mathbb{Z}}$ of abelian groups. It is then natural to ask whether the entire theory of abelian categories can be described entirely in terms of this universal abelian category. In homological algebra, we have the classical Freyd–Mitchell embedding theorem (cf. [Gin05]).

Theorem 4.2 (Freyd–Mitchell). *Let $\mathcal{A}$ be an abelian category. Then there exists a ring R and a fully faithful exact functor*

$$\mathcal{A} \hookrightarrow \mathrm{Mod}_R \simeq \mathrm{Mod}(\mathrm{Mod}_{\mathbb{Z}}; R)$$

[3]Note that $\mathcal{S}$ is not pointed yet, but we omit the basepoint from the notation $\mathrm{Sp}(\mathcal{S})$ since it is obvious that one first needs to make this into a pointed presentable ∞-category before stabilising.

embedding $\mathcal{A}$ exactly into the abelian category of R-modules. Further, this embedding is essentially surjective ergo an equivalence if $\mathcal{A}$ has all small coproducts and admits a compact projective generator C. In fact, one can identify

$$R \cong \mathrm{End}_{\mathcal{A}}(C) \in \mathrm{Alg}(\mathrm{Mod}_{\mathbb{Z}}).$$

We see that the theory of abelian categories can therefore be subsumed into the theory of module categories, where the ring that these modules are taken over even admits an explicit description. Recall that a compact projective object in an (∞-)category is an object c such that the functor $\mathrm{map}(c, -)$ commutes with sifted colimits.

The result above follows from a certain monadicity result, known as the Barr–Beck theorem, or Barr–Beck–Lurie in the ∞-categorical case. The goal of this section will be to state a similar result in the theory of presentable stable (i.e. Sp-linear) ∞-categories, and end with a generalisation to $\mathcal{C}$-linear ∞-categories, where $\mathcal{C}$ is of the form

$$\mathcal{C} = \mathcal{F}\mathrm{un}(K^{\mathrm{op}}, \mathrm{Sp}).$$

Let us now present the main workhorse of this section, which is the Barr–Beck–Lurie theorem from [Lur17] Theorem 4.7.3.5.

Theorem 4.3 (Barr–Beck–Lurie)**.** *Consider an adjunction of ∞-categories*

$$f^* : \mathcal{C} \rightleftarrows \mathcal{D} : f_*$$

and its associated monad

$$T = f_* f^* \in \mathrm{Alg}(\mathrm{End}(\mathcal{C})).$$

There is a natural factorisation

$$f^* : \mathcal{C} \rightleftarrows \mathrm{Mod}(\mathcal{C}; T) \underset{\tilde{f}_*}{\overset{\tilde{f}^*}{\rightleftarrows}} \mathcal{D} : f_*$$

through the category of modules over T with its associated free/forgetful adjunction. The following are equivalent

- *The adjunction $\tilde{f}^* \dashv \tilde{f}_*$ is an equivalence, i.e. $f^* \dashv f_*$ is a monadic equivalence.*
- *$\mathcal{D}$ admits geometric realisations, and they are preserved by f_*, which is furthermore conservative.*

Let us illustrate this with a pair of examples.

Example 4.1. Suppose that $\mathcal{C}$ and $\mathcal{D}$ are both $\mathbb{E}_n$-algebras in $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}$, i.e. stable presentably $\mathbb{E}_n$-monoidal ∞-categories, and that the left adjoint f^* is a strict $\mathbb{E}_n$-monoidal functor. Then the right adjoint f_* admits the structure of a lax $\mathbb{E}_n$-monoidal functor. In particular, it sends the $\mathbb{E}_n$-algebra $\mathbb{1}_{\mathcal{D}}$ to an $\mathbb{E}_n$-algebra $f_*\mathbb{1}_{\mathcal{D}}$ in $\mathcal{C}$. The monad T can then be identified with the operation $f_*\mathbb{1}_{\mathcal{D}} \otimes -$

Example 4.2. Let $\mathcal{C}$ be an element of $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}$. Then a choice of compact object $c : \Delta^0 \to \mathcal{C}$ can be extended to a left adjoint

$$c^* : \mathrm{Sp} \simeq \mathrm{Sp}(\mathcal{P}(\Delta^0)) \to \mathcal{C}$$

by Remark 4.4. This has a right adjoint given by

$$c_* = \mathrm{Map}(c, -) : \mathcal{C} \to \mathrm{Sp},$$

i.e. taking the mapping spectrum out of c in the stable ∞-category $\mathcal{C}$. The monad T can be identified with

$$T = c_* c^*(x) = \mathrm{Map}(c, c),$$

so that the Barr–Beck–Lurie theorem tells us that this adjunction is monadic if and only if $\mathrm{Map}(c, -)$ is conservative and preserves simplicial colimits, i.e. c is compact, and is a generator of $\mathcal{C}$.

This last remark is precisely our Sp-linear analogue of the Freyd-Mitchell embedding theorem, namely the Schwede–Shipley recoginition theorem, cf. [Lur17] Theorem 7.1.2.1.

Theorem 4.4 (Schwede–Shipley)**.** *Let $\mathcal{C}$ be an element of $\mathcal{P}r^{\mathrm{L}}_{\mathrm{St}}$ which is generated under colimits by a compact projective object c. We will refer to this type of stable homotopy theory as monogenic in reference to [HPS97]. Then the adjunction*

$$c^* : \mathrm{Sp} \rightleftarrows \mathcal{C} : c_*$$

is monadic, and one can identify the monad c_*c^* with the $\mathbb{E}_1$-ring $\mathrm{Map}_{\mathcal{C}}(c, c)$ in Sp, whence one obtains an equivalence

$$\mathcal{C} \simeq \mathrm{Mod}(\mathrm{Sp}; \mathrm{End}_{\mathcal{C}}(c)).$$

Remark 4.6. We remark that if $\mathcal{C}$ is symmetric monoidal and such that c is the unit $\mathbb{1}_{\mathcal{C}}$, then this equivalence can be extended to a symmetric monoidal equivalence.

This result is extremely useful in homotopy theory, since it precisely tells us that sufficiently nice homotopy theories (i.e. those generated by a singe compact projective object) can be ensconced within the homotopy theory of spectra, but with a module structure over some rather complicated algebra object encapsulating the structure of the original stable homotopy theory. This program can be extended to more general cases where we replace Δ^0 with an arbitrary grouplike monoidal ∞-category K, hence relaxing the monogenic condition. This process is described in [Heg18], and it is our main reference for this section.

Definition 4.11. A grouplike symmetric monoidal ∞-category K is a monoidal ∞-category such that every element is invertible for the monoidal structure. Our primary examples are the discrete ∞-category $\mathbb{Z}^\delta$ and the poset $\mathbb{Z}$ (these will be discussed in more detail in Section 5) with monoidal structure given by addition.

Given such a K, we consider the ∞-category

$$\mathrm{Sp}^{K^{\mathrm{op}}} = \mathcal{F}\mathrm{un}(K^{\mathrm{op}}, \mathrm{Sp}) \simeq \mathrm{Sp}(\mathcal{P}(K)),$$

viewed as an element of $\mathrm{CAlg}(\mathcal{P}r^{\mathrm{L}}_{\mathrm{St}})$ using the Day convolution monoidal structure. In particular, one can consider the ∞-category

$$\mathrm{Mod}(\mathcal{P}r^{\mathrm{L}}_{\mathrm{St}}; \mathrm{Sp}^{K^{\mathrm{op}}})$$

of $\mathcal{F}\mathrm{un}(K^{\mathrm{op}}, \mathrm{Sp})$-linear ∞-categories. These are enriched in $\mathrm{Sp}^{K^{\mathrm{op}}}$ by a direct application of [Lur17] Proposition 4.2.1.33. One can think of these enriched mapping objects as arising from the usual left adjoint

$$c_* : \mathrm{Sp} \to \mathcal{C}$$

induced by the functor $\Delta^0 \to \mathcal{C}$ picking out an object c. The right adjoint of this functor is precisely the mapping spectrum functor $\mathrm{Map}(c, -)$. We can then use the fact that the target is an $\mathrm{Sp}^{K^{\mathrm{op}}}$-module to tensor this up to a left adjoint

$$c_*^K : \mathrm{Sp} \otimes \mathrm{Sp}^{K^{\mathrm{op}}} \simeq \mathrm{Sp}^{K^{\mathrm{op}}} \to \mathcal{C},$$

whose right adjoint is precisely $\mathrm{Map}^K(c, -)$.

We now proceed to give an $\mathrm{Sp}^{K^{\mathrm{op}}}$-linear analogue of the Schwede–Shipley theorem.

Theorem 4.5. *Let $\mathcal{C}$ be an element of $\mathrm{Mod}(\mathcal{P}r^{\mathrm{L}}_{\mathrm{St}}; \mathrm{Sp}^{K^{\mathrm{op}}})$ which contains some compact object c such that*

$$\mathrm{Map}^K(c, -) : \mathcal{C} \to \mathrm{Sp}^{K^{\mathrm{op}}}$$

is a conservative. We will refer to this type of stable homotopy theory as K-plurigenic. Then the adjunction

$$\mathrm{Sp}^{K^{\mathrm{op}}} \rightleftarrows \mathcal{C}$$

is monadic, and one can identify the monad with the $\mathbb{E}_1$-ring $\mathrm{Map}^K(c, c)$ in $\mathrm{Sp}^{K^{\mathrm{op}}}$, whence one obtains an equivalence

$$\mathcal{C} \simeq \mathrm{Mod}(\mathrm{Sp}^{K^{\mathrm{op}}}; \mathrm{End}^K_{\mathcal{C}}(c)).$$

This theorem is stated here as in Theorem 2.15 of [Heg18]. For completeness, we will reproduce the proof in op. cit. Note that it is a straightforward generalisation of the proof of the Schwede–Shipley theorem in [Lur17] Theorem 7.1.2.1

Proof. This proof essentially follows from a reduction to Proposition 4.5.8.5. in op. cit, using the observation that $\mathcal{C}$ is an $\mathrm{Sp}^{K^{\mathrm{op}}}$-module in $\mathcal{P}r^{\mathrm{L}}_{\mathrm{St}}$, hence left tensored over the latter. It suffices to show the following statements:

1. $\mathcal{C}$ admits geometric realisations,
2. and these are preserved by the right adjoint $G : \mathcal{C} \to \mathrm{Sp}^{K^{\mathrm{op}}}$ to left tensoring map at c
$$F : \mathrm{Sp}^{K^{\mathrm{op}}} \times \mathcal{C} \to \mathcal{C} : a \mapsto a \otimes c.$$
3. The left tensoring map
$$\mathrm{Sp}^{K^{\mathrm{op}}} \times \mathcal{C} \to \mathcal{C}$$
should preserve geometric realisations.
4. G should be conservative.
5. G and F should satisfy a sort of projection formula, stating that the natural map
$$F(a \otimes G(x)) = (a \otimes G(x)) \otimes c \simeq a \otimes (G(x) \otimes c) = a \otimes F(G(x)) \to a \otimes x$$
ought to be adjoint along $F \dashv G$ to an equivalence
$$a \otimes G(x) \xrightarrow{\sim} G(a \otimes x).$$

Now the presentability assumption already buys us a lot. Indeed, since $\mathrm{Sp}^{K^{\mathrm{op}}}$ and $\mathcal{C}$ are presentable and $\mathcal{C}$ is an $\mathrm{Sp}^{K^{\mathrm{op}}}$-module in $\mathcal{P}r^{\mathrm{L}}_{\mathrm{St}}$, i.e. such that all structure maps are left adjoints, we immediately see that point 1 holds. Further, $\mathcal{C}$ is assumed to be presentably symmetric monoidal, so that one can apply the adjoint functor theorem to F to obtain the existence of G, as well as note that 3 holds. To prove point 2, we note that F is left adjoint to the enriched mapping functor $\mathrm{Map}^K(c, -)$ per construction. In particular, this means that G admits an explicit description as such. Since c was assumed to be compact, we see that G then commutes with filtered colimits. By the stability assumption, we see that it also commutes with cofibre sequences sine these are fibre sequences. We conclude that G is cocontinuous, hence preserves geometric realisations. The fact that G is conservative is then precisely the condition we required c to satisfy. Finally, let us show that the map $a \otimes G(x) \to G(a \otimes x)$ is an equivalence for all $a \in \mathrm{Sp}^{K^{\mathrm{op}}}$. Since this ∞-category is (the stabilisation of) a presheaf category on K, it suffices to check this for all generators $a = \Sigma^{\infty+n} よ(k), k \in K$. Indeed, G is cocontinuous and both $\mathrm{Sp}^{K^{\mathrm{op}}}, \mathcal{C}$ are presentably symmetric monoidal, so that expressing a as a colimit of these representables the result would follow. Once again using cocontinuity we can assume $n = 0$. But now this becomes tautological. Indeed, G is $\mathrm{Sp}^{K^{\mathrm{op}}}$-linear in the sense that the left tensoring of $\mathcal{C}$ over $\mathrm{Sp}^{K^{\mathrm{op}}}$ is defined such that

$$\begin{aligned} \Sigma^\infty よ(k) \otimes \mathrm{Map}^K(c, x) &\simeq \Sigma^\infty よ(k) \otimes (k' \mapsto \mathrm{Map}(c, \Sigma^\infty よ(k') \otimes x)), \\ &\simeq (k' \mapsto \mathrm{Map}(c, \Sigma^\infty よ(k) \otimes \Sigma^\infty よ(k') \otimes x), \\ &\simeq \mathrm{Map}^K(c, \Sigma^\infty よ(k) \otimes x). \end{aligned}$$

□

Remark 4.7. By a similar observation as before, we see that this equivalence is symmetric monoidal if c is the unit, by applying the universal property of Day convolution.

Remark 4.8. This result is clearly a generalisation of the classical result of Schwede–Shipley when $K = \Delta^0$, and a K-plurigenic homotopy theory is easily seen to be a monogenic homotopy theory. The difference then lies in the K^{op}-indexed mapping spectra, that keep track not only of the compact object in $\mathcal{C}$ but also all of its shifts along morphisms in K. This admits a particularly simple description when $K = \mathbb{Z}^\delta$ or $\mathbb{Z}$, and the latter will be discussed extensively in the description of deformations of stable homotopy theories.

4.5 Dwyer–Greenlees theory in $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}$

To describe recollements of stable presentably symmetric monoidal ∞-categories, especially in the case where this recollement arises from a deformation, it will be useful to develop the arithmetic of stable presentably symmetric monoidal ∞-categories. In this section, we will describe how to decompose these into complete or torsion and invertible objects with respect to certain elements. The classical analogy for this is the decompostion of abelian groups into $\mathbb{Z}[p^{-1}]$-modules and p-complete abelian groups; leading to the usual fracture square. As we will see later on, this is a special case of a recollement, and this section aims to introduce some of the most salient examples of these.

The primary results of this section reflect those of [DG02], in which they define trivial, torsion, and complete objects in certain enriched categories of cochain complexes, defined in terms of the contractibility of certain mapping complexes. As claimed in op. cit. this theory extends to stable presentably symmetric monoidal ∞-categories, as expounded in detail in our primary reference for this section, [MNN17].

The ingredients are the following:

- A stable and presentably symmetric monoidal ∞-category $\mathcal{C}$, such as the ∞-category of spectra, filtered spectra, or synthetic spectra (to be discussed below).
- A homotopy associative algebra object A, such as an Adams type homology theory in spectra, or $C\tau$ in filtered/synthetic spectra (see below).

We impose the following conditions, which are all met by our primary examples of interest.

- We assume that $\mathcal{C}$ is generated by dualisable objects, and that the monoidal unit $\mathbb{1}_{\mathcal{C}}$ is compact (whence all dualisables are compact as well).
- We assume that the underlying object of A is dualisable.

Definition 4.12. Given such an A, we can form a family of full subcategories of $\mathcal{C}$.

- First, define the A-**trivial** objects to be the $T \in \mathcal{C}$ such that
$$T \otimes A \simeq 0.$$
- Their right orthogonal complement, i.e. the $X \in \mathcal{C}$ such that for all A-trivial T we have
$$\mathrm{map}(T, X) \simeq 0$$
are called A-**complete**. They form a full subcategory $\mathcal{C}^\wedge_A$. Note from this definition that one can identify $\mathcal{C}^\wedge_A$ with the Bousfield localisation of $\mathcal{C}$ at the $-\otimes A$-equivalences.
- The smallest localising subcategory of $\mathcal{C}$ containing the objects
$$A \otimes D$$
for $D \in \mathcal{C}$ dualisable is the subcategory of A-**torsion** objects ${}_A\mathcal{C}$.
- Their right orthogonal complement, i.e. the $Y \in \mathcal{C}$ such that for all A-torsion S we have
$$\mathrm{map}(S, Y) \simeq 0$$
are called A-**invertible**. They form a full subcategory $\mathcal{C}[A^{-1}]$

These definitions admit some immediate observations about reflectivity properties of these subcategories.

Remark 4.9. The subcategories $\mathcal{C}^\wedge_A$ and $\mathcal{C}[A^{-1}]$ are defined in terms of mapping spaces into them, which clearly commutes with limits. Therefore, these subcategories are closed under limits, whence they form reflective subcategories of $\mathcal{C}$. Indeed, using presentability, this guarantees the existence of A-completion and A-inversion functors, which are the reflections back into these subcategories. On the other hand, colimits in the first two subcategories are computed by taking colimits in the ambient ∞-category $\mathcal{C}$, and then applying the A-completion and A-inversion functors respectively. Dually, the inclusion of the A-torsion objects ${}_A\mathcal{C} \subset \mathcal{C}$ admits a right adjoint.

We now come to the main result of this section, which is an identification between the A-torsion and A-complete objects of $\mathcal{C}$.

Lemma 4.1 ([MNN17] Theorem 3.9). *The functor $(-)^\wedge_A : \mathcal{C} \to \mathcal{C}^\wedge_A$ restricts to an equivalence*

$$ {}_A\mathcal{C} \simeq \mathcal{C}^\wedge_A. $$

Proof. This follows from the definitions above, making crucial use of the assumption that A is dualisable, and is discussed in detail in op. cit. □

A particularly interesting feature of this is that the A-complete objects–forming a reflective subcategory–are identified with the A-torsion objects, which form a coreflective subcategory. In particular, this carries over dually to the right orthogonal complement of the latter–namely the full subcategory of A-invertible objects. We conclude that the inclusions and adjoints to them lie in a commutative diagram

$$\begin{tikzcd}
 & & {}_A\mathcal{C} \arrow[dl, bend right, hook'] \arrow[dd, bend left, "\sim"] \\
\mathcal{C}[A^{-1}] \arrow[r, hook] & \mathcal{C} \arrow[l, bend right, "A^{-1}"'] \arrow[l, bend left] \arrow[ur, "{}_A(-)"'] \arrow[dr, "(-)^\wedge_A"] & \\
 & & \mathcal{C}^\wedge_A \arrow[ul, bend left, hook']
\end{tikzcd}$$

where all horizontal or diagonal arrows are left adjoint to the one lying below them. The two inclusions on the right hand side are the inclusions of A-complete and A-torsion objects, respectively right and left adjoint to the A-completion functor. By commutativity, it is often easier to leave the identification ${}_A\mathcal{C} \simeq \mathcal{C}^\wedge_A$ implicit, and write this diagram in the form

$$\begin{tikzcd}
\mathcal{C}[A^{-1}] \arrow[r, hook] & \mathcal{C} \arrow[l, bend right=50, "A^{-1}"'] \arrow[l, bend left=50] \arrow[r, "(-)^\wedge_A"] & \mathcal{C}^\wedge_A . \arrow[l, bend right=50, hook'] \arrow[l, bend left=50, hook']
\end{tikzcd}$$

We will see later that this diagram is important as it describes a recollement of $\mathcal{C}$.

Remark 4.10. Now let us provide some characterisations of complete objects. Note that any A-module is clearly A-complete. Indeed for T A-trivial, the universal property of free A-modules gives

$$ \mathrm{map}(T, X) \simeq \mathrm{map}(T \otimes A, X) \simeq \mathrm{map}(0, X) \simeq 0. $$

Lemma 4.2. *The A-completion of an object of X admits an explicit description in terms of the cobar resolution of A. Explicitly, the unit map $X \longrightarrow X^\wedge_A$ can be identified with the augmentation map*

$$ X \to \lim_{\Delta} X \otimes A^{\otimes \bullet}. $$

Proof. This builds on the previous remark, by nothing that the cobar resolution of X is akin to a resolution of X by A-modules. The latter are all complete, and the subcategory of A-complete objects is closed under limits, whence we see that the target of the augmentation map is actually an element of $\mathcal{C}^\wedge_A$. Now note that A is dualisable, so that $A \otimes -$ preserves limits–being right adjoint to tensoring with the dual of A. We therefore see that the augmentation map can be tensored with A to obtain a map

$$ X \otimes A \to \lim_{\Delta} \; X \otimes A^{\otimes \bullet + 1}. $$

Now the cosimplicial object inside the limit is just a split cosimplicial object and the map is its augmentation, so that it must be an equivalence. We conclude that the original augmentation map is a $- \otimes A$-equivalence to an A-complete object, hence ought to be seen as an A-completion map. □

5 Filtered spectra

In this section, we introduce the ∞-category of *filtered spectra*, whose relation with the ∞-category of spectra will play a fundamental rôle in the yoga of deformations of homotopy theories. A related notion is that of graded spectra, which will be introduced as well.

Consider the poset of integers $\mathbb{Z}$, with the poset structure being given by the usual order relation $\leq$, and view this as a $(1,1)$-category in the obvious way. This is further equipped with a symmetric monoidal structure given by addition of integers, so that its nerve is a symmetric monoidal ∞-category, also denoted $\mathbb{Z}$. On the other hand, we can also view the integers $\mathbb{Z}$ as a monoid without its order, and view this as a discrete groupoid in spaces. The resulting symmetric monoidal ∞-category is denoted $\mathbb{Z}^\delta$. These two ∞-categories allow us to define filtered and graded objects respectively.

Definition 5.1. Let $\mathcal{C}$ be some ∞-category, which will always be stable and presentably symmetric monoidal in the future. Then a **filtered object** of $\mathcal{C}$ is a functor $\mathbb{Z}^{\mathrm{op}} \to \mathcal{C}$, while a **graded object** of $\mathcal{C}$ is a functor $\mathbb{Z}^\delta \to \mathcal{C}$. These are the objects of the ∞-categories of filtered and graded objects in $\mathcal{C}$ respectively, which we define as the functor ∞-categories

$$\mathcal{C}^{\mathrm{Fil}} = \mathcal{F}\mathrm{un}(\mathbb{Z}^{\mathrm{op}}, \mathcal{C}), \qquad\qquad \mathcal{C}^{\mathrm{Gr}} = \mathcal{F}\mathrm{un}(\mathbb{Z}^{\delta}, \mathcal{C}).$$

Note that these ∞-categories are once again symmetric monoidal by equipping them with the Day convolution. Letting K be some indexing ∞-category with symmetric monoidal structure denoted $\otimes_K$, and letting $\otimes_{\mathcal{C}}$ denote the symmetric monoidal structure on $\mathcal{C}$, recall that the the Day convolution of two functors $X, Y : K^{\mathrm{op}} \to \mathcal{C}$ is defined by

$$(X \otimes Y)_k = \operatorname*{colim}_{k \to k' \otimes_K k''} X_{k'} \otimes_{\mathcal{C}} X_{k''}.$$

When K is given by $\mathbb{Z}$ or $\mathbb{Z}^{\mathrm{op}}$ this can be slightly rephrased by noting that the former is a poset while the latter is discrete hence does not have any nontrivial morphisms.

$$\begin{aligned} \otimes : \mathcal{C}^{\mathrm{Fil}} \times \mathcal{C}^{\mathrm{Fil}} &\to \mathcal{C}^{\mathrm{Fil}}, \\ X, Y \mapsto X \otimes Y &: n \mapsto \operatorname*{colim}_{n \leq p+q} X_p \otimes_{\mathcal{C}} Y_q, \end{aligned} \qquad\qquad \begin{aligned} \otimes : \mathcal{C}^{\mathrm{Gr}} \times \mathcal{C}^{\mathrm{Gr}} &\to \mathcal{C}^{\mathrm{Gr}}, \\ X, Y \mapsto X \otimes Y &: n \mapsto \bigoplus_{n \leq p+q} X_p \otimes_{\mathcal{C}} Y_q. \end{aligned}$$

In particular, this allows us to describe the unit of $\mathcal{C}^{\mathrm{Fil}}$ as the filtered object given by

$$\mathbb{1}_{\mathcal{C}^{\mathrm{Fil}}} = \cdots \to 0 \to 0 \to \mathbb{1}_{\mathcal{C}} \to \mathbb{1}_{\mathcal{C}} \to \cdots,$$

which is zero in degrees ≥ 1 and consists of the unit of $\mathcal{C}$ and identity maps in all lower degrees. Indeed, it is easy to verify that for any other filtered object X of $\mathcal{C}$ we have

$$\begin{aligned} (X \otimes \mathbb{1}_{\mathcal{C}^{\mathrm{Fil}}})_n &= \operatorname*{colim}_{n \leq p+q} X_p \otimes_{\mathcal{C}} \mathbb{1}_{\mathcal{C}^{\mathrm{Fil}}\, q}, \\ &\simeq \operatorname*{colim}_{n \leq p+q,\, q \leq 0} X_p[q]. \end{aligned}$$

The diagram formed by these $X_p[q]$ looks like a slice in the fourth and third quadrants of the (p,q)-plane. Recall that the arrows between the objects go in the direction of decreasing p, q. Diagramatically,

$$\begin{tikzcd} X_n, & X_{n+1} \arrow[l] \arrow[d] & X_{n+2} \arrow[l] \arrow[d] & X_{n+3} \arrow[l] \arrow[d] & \cdots \arrow[l] \\ & X_{n+1} & X_{n+2} \arrow[l] \arrow[d] & X_{n+3} \arrow[l] \arrow[d] & \cdots \arrow[l] \\ & & X_{n+2} & X_{n+3} \arrow[l] \arrow[d] & \cdots \arrow[l] \\ & & & X_{n+3} & \cdots \arrow[l] \end{tikzcd}$$

it becomes clear that the down-and-left direction of the arrows allows us to distinguish a cofinal subdiagram consisting of all objects sitting on the outermost diagonal $p+q=n$ and the spans formed with objects on the diagonal $p+q=n+1$. Now note that all vertical morphisms in consideration were obtained by tensoring with the identity of $\mathbb{1}_{\mathcal{C}}$, hence are invertible. This allows us to consider an equivalent cofinal subdiagram where the downward arrows now point up. It is clear that this has a terminal object, namely X_n. We conclude that the colimit in the expression for $(X \otimes \mathbb{1}_{\mathcal{C}^{\mathrm{Fil}}})_n$ is simply X_n itself. Further, the connecting maps between filtration degrees are clearly equivalent to the original structure maps in X itself.

As mentioned above, the base ∞-category $\mathcal{C}$ will always be assumed to be stable and presentably symmetric monoidal, i.e. an element of $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}})$. Now the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}$ has a tensor unit given by the ∞-category of spectra Sp, so that many results about homotopy theories can be proven in spectra and then induced up to more general homotopy theories. There is a similar situation for the theories of filtered and graded objects, where results about filtered and graded spectra can be tensored up to more general settings:

Lemma 5.1. *Let K be an ∞-category, and consider the functor ∞-category $\mathcal{F}\mathrm{un}(K^{\mathrm{op}}, \mathcal{C})$ of K-shaped diagrams in some stable, presentably symmetric monoidal ∞-category $\mathcal{C}$. Then there is an equivalence*

$$\mathcal{F}\mathrm{un}(K^{\mathrm{op}}, \mathcal{C}) \simeq \mathcal{F}\mathrm{un}(K^{\mathrm{op}}, \mathrm{Sp}) \otimes \mathcal{C}.$$

Proof. This follows from the definitions and the universal property of stabilisation and presheaves. Indeed the right hand side is defined as

$$\mathcal{F}\mathrm{un}(K^{\mathrm{op}}, \mathrm{Sp}) \otimes \mathcal{C} := \mathcal{F}\mathrm{un}^{\mathrm{R}}(\mathcal{F}\mathrm{un}(K^{\mathrm{op}}, \mathrm{Sp})^{\mathrm{op}}, \mathcal{C}).$$

We then note that

$$\begin{aligned} \mathcal{F}\mathrm{un}^{\mathrm{R}}(\mathcal{F}\mathrm{un}(K^{\mathrm{op}}, \mathrm{Sp})^{\mathrm{op}}, \mathcal{C}) &\simeq \mathcal{F}\mathrm{un}^{\mathrm{L}}(\mathcal{F}\mathrm{un}(K^{\mathrm{op}}, \mathrm{Sp}), \mathcal{C}^{\mathrm{op}})^{\mathrm{op}}, \\ &\simeq \mathcal{F}\mathrm{un}^{\mathrm{L}}(\mathrm{Sp}(\mathcal{P}(K)), \mathcal{C}^{\mathrm{op}})^{\mathrm{op}}, \\ &\simeq \mathcal{F}\mathrm{un}(K, \mathcal{C}^{\mathrm{op}})^{\mathrm{op}}, \\ &\simeq \mathcal{F}\mathrm{un}(K^{\mathrm{op}}, \mathcal{C}), \end{aligned}$$

where we used the duality of limits and colimits to recover the setting of equivalence 1, further using the enriched adjunction formula in Remark 4.4. While it is clear that $\mathcal{C}^{\mathrm{op}}$ is stable if and only if $\mathcal{C}$ is, to apply the universal property of presheaf categories we used the fact that any presentable ∞-category also has all small limits, which is a rather immediate result of the presentation of the latter as localisations of presheaf ∞-categories, cf. [Lur09] 5.5.2.4. This guarantees that $\mathcal{C}^{\mathrm{op}}$ is cocomplete. Further, we see that this equivalence is symmetric monoidal by the universal property of the Day convolution as left Kan extending the monoidal structure on K along the Yoneda embedding and stabilisation. □

Remark 5.1. When K is symmetric monoidal and $\mathcal{F}\mathrm{un}(K^{\mathrm{op}}, \mathrm{Sp})$ is given the Day convolution structure, we see that this Lemma is simply telling us that one can go from Sp-linear category theory (i.e. working in $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}$) to $\mathrm{Sp}^{K^{\mathrm{op}}}$-linear category theory (i.e. working in $\mathrm{Mod}(\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}; \mathrm{Sp}^{K^{\mathrm{op}}})$) by forming free objects obtained by tensoring with $\mathrm{Sp}^{K^{\mathrm{op}}}$, and that the latter admit a simple description as $\mathcal{C}^{K^{\mathrm{op}}}$. Further, if $\mathcal{C}$ was an Sp-algebra, then it is clear from the comment above that the equivalence

$$\mathrm{Sp}^{K^{\mathrm{op}}} \to \mathcal{C}^{K^{\mathrm{op}}}$$

is monoidal for the Day convolution on both sides.

In particular, setting $K = \mathbb{Z}$, resp. $K = \mathbb{Z}^{\delta}$, we obtain[4]

$$\mathcal{C}^{\mathrm{Fil}} \simeq \mathrm{Sp}^{\mathrm{Fil}} \otimes \mathcal{C}, \qquad \mathcal{C}^{\mathrm{Gr}} \simeq \mathrm{Sp}^{\mathrm{Gr}} \otimes \mathcal{C}.$$

Now that the theory of filtered and graded objects has been set up and we have observed that we can safely restrict to spectra, let us describe a few important adjunctions between $\mathrm{Sp}^{\mathrm{Fil}}$, $\mathrm{Sp}^{\mathrm{Gr}}$, and Sp.

[4]We will always identify $\mathbb{Z}^{\delta}$ with $(\mathbb{Z}^{\delta})^{\mathrm{op}}$ to lighten notation.

Definition 5.2.

- Since Sp is the unit of $\mathcal{P}r^{L}_{St}$, there is an essentially unique symmetric monoidal left adjoint $\mathrm{Sp} \to \mathrm{Sp}^{\mathrm{Fil}}$. This functor will be denoted by c. It is easily verified that this is given by

$$cX_\star = \begin{cases} 0, & \star \geq 1, \\ X, & \star \leq 0, \end{cases}$$

  with the connecting morphisms being either zero maps or identities.

- There is another functor from Sp to $\mathrm{Sp}^{\mathrm{Fil}}$ which can be viewed as some sort of trivial filtration. It is denoted[5]

$$\mathrm{Cs} : \mathrm{Sp} \to \mathrm{Sp}^{\mathrm{Fil}}$$

  and defined by sending some spectrum X to the filtered spectrum consisting of X in every degree and identity maps between them. In fact, this is just the constant diagram functor from Sp to the diagram category $\mathcal{F}\mathrm{un}(\mathbb{Z}^{\mathrm{op}}, \mathrm{Sp})$. Therefore, we immediately note that Cs has left resp. right adjoints given by the colimit resp. limit of a filtered spectrum along the indexing category $\mathbb{Z}^{\mathrm{op}}$. The former of these will be referred to as the realisation functor Re, while the latter is called the intersection In. We therefore obtain an adjunction

$$\begin{tikzcd} \mathrm{Sp} \arrow[r, "\mathrm{Cs}"] & \mathrm{Sp}^{\mathrm{Fil}} \arrow[l, bend right, "\mathrm{Re}"'] \arrow[l, bend left, "\mathrm{In}"] \end{tikzcd}$$

$$\mathrm{Re}(X_\star) = \operatorname*{colim}_{n \in \mathbb{Z}^{\mathrm{op}}} X_n, \qquad \mathrm{Cs}(X) = \cdots \to X \xrightarrow{\mathrm{id}} X \to \cdots, \qquad \mathrm{In}(X_\star) = \lim_{n \in \mathbb{Z}^{\mathrm{op}}} X_n.$$

- Note that there is an obvious inclusion $\mathbb{Z}^\delta \to \mathbb{Z}^{\mathrm{op}}$ of symmetric monoidal categories. Precomposition with this inclusion then gives us a functor

$$U : \mathrm{Sp}^{\mathrm{Fil}} \to \mathrm{Sp}^{\mathrm{Gr}}$$

  that sends a filtered spectrum $X_\star$ simply to its collection of objects $\{X_n\}_{n \in \mathbb{Z}}$ without the connecting homomorphisms. Since the categories $\mathrm{Sp}^{\mathrm{Fil}}$ and $\mathrm{Sp}^{\mathrm{Gr}}$ are presentable, we can find a left adjoint to this precomposition by left Kan extension. It is easily verified that the the left Kan extension functor $L : \mathrm{Sp}^{\mathrm{Gr}} \to \mathrm{Sp}^{\mathrm{Fil}}$ is defined by

$$L(X_*)_n = \bigoplus_{m \leq n} X_m \to \bigoplus_{m \leq n-1} X_m = L(X_*)_{n-1},$$

  i.e. by progressively smaller direct sums and projections between them.

- Finally, we come to a functor which is important in the description of filtered spectra as filtrations, namely the associated graded functor. This is defined by

$$\mathrm{gr} : \mathrm{Sp}^{\mathrm{Fil}} \to \mathrm{Sp}^{\mathrm{Gr}} : \mathrm{gr}(X_\star)_n = X_n / X_{n+1},$$

  where the notation X_n/X_m for $m \geq n$ refers to the cofibre of the essentially uniquely prescribed map $X_m \to X_n$. Note that colimits in diagram categories are computed levelwise, so that the levelwise cofibre expression of gr tells us that it preserves colimits. We conclude that it is a left adjoint by presentability of Sp. It is also straightforward to check that it is strict symmetric monoidal with respect to the Day convolution on both sides. Indeed, it is immediate to verify this on objects of the form $\mathbb{1}_{\mathrm{Fil}}(n)$, and since these generate $\mathrm{Sp}^{\mathrm{Fil}}$ under colimits and gr is a left adjoint, the result follows. This is worked out in [Hed20] Proposition II.1.13.

[5] If we were to follow an analogy with the theory of synthetic spectra of [Pst18] in naming this, it would be called Y like the spectral Yoneda embedding. However, this notation might cause confusion.

As the name implies, a filtered spectrum $X_\star : \mathbb{Z}^{\mathrm{op}} \to \mathrm{Sp}$ ought to be interpreted as a descending filtration of its realisation $\mathrm{Re}(X_\star)$. This is why filtered spectra are relevant objects, since they allow us to resolve spectra by more familiar of computationally tractable objects. In particular, we single out a few examples of filtrations that arise naturally.

Example 5.1.

1. Consider the usual t-structure on Sp with truncation functors
$$\tau_{\geq n} : \mathrm{Sp} \to \mathrm{Sp}_{\geq n} \subset \mathrm{Sp}.$$
Using the canonical natural transformations
$$\tau_{\geq n} \longrightarrow \tau_{\geq n-1},$$
these assemble to a functor
$$\tau_{\geq \star} : \mathrm{Sp} \to \mathrm{Sp}^{\mathrm{Fil}}$$
in the obvious way. In fact, this construction is lax symmetric monoidal as mentioned in [BHS20] Example C.5. It is also clear that this construction extends to arbitrary stably symmetric monoidal ∞-categories with a t-structure that is compatible with the symmetric monoidal structure. In fact, left and right completeness of the Postnikov t-structure on Sp guarantee that this filtered spectrum has is complete and realises to the original underlying spectrum.

2. Given a lax symmetric monoidal functor $T : \mathrm{Sp} \to \mathrm{Sp}^{\mathrm{Fil}}$, and an $\mathbb{E}_n$-algebra E in Sp, we obtain a functor called the *décalage* of T along E from spectra to filtered spectra. It is defined by sending some X to the cosimplicial spectrum obtained by tensoring with the cobar resolution $E^{\wedge\bullet}$ of E, then applying T to obtain a filtered cosimplicial spectrum (it is clear that this is equivalent to a cosimplicial filtered spectrum). Finally, applying the totalisation (i.e. limit over Δ) to this cosimplicial object, we obtain a filtered spectrum denoted $\mathrm{Déc}(T;E)(X)$. In terms of formulas:
$$\mathrm{Déc}(T;E) : \mathrm{Sp} \xrightarrow{\wedge E^{\wedge\bullet}} \mathrm{Sp}^{\Delta} \xrightarrow{T} (\mathrm{Sp}^{\mathrm{Fil}})^{\Delta} \xrightarrow{\mathrm{Tot}} \mathrm{Sp}^{\mathrm{Fil}}.$$
Now smashing with the cobar contruction is lax symmetric monoidal by virtue of the the multiplication maps of E giving rise to a map
$$X \wedge E^{\wedge n} \wedge Y \wedge E^{\wedge n} \simeq X \wedge Y \wedge E^{\wedge 2n} \to X \wedge Y \wedge E^{\wedge n}$$
for any spectra X, Y and $n \geq 0$. Since the multiplication maps are precisely the coface maps of $E^{\wedge\bullet}$, these assemble to a map of cosimplicial objects making $- \wedge E^{\wedge\bullet}$ lax monoidal as required.

   Since T was assumed to be lax monoidal, we see that the composite functor $\mathrm{Déc}(T;E)$ is lax monoidal. This means that it sends $\mathbb{E}_n$-algebras in spectra to $\mathbb{E}_n$-algebras in filtered spectra. In fact, this construction is essential for constructing examples of the latter.[6]

5.1 The thread operator

In this section, we show that $\mathrm{Sp}^{\mathrm{Fil}}$ has a certain endofunctor which plays a very important rôle in the deformation picture. First, note that any filtered spectrum $X_\star$ can be shifted, i.e. we can define a new filtered spectrum $X(m)_\star$ by $X(m)_\star = X_{\star-m}$ with the obvious morphisms. In fact, one can easily verify the formula $X(n) \simeq X \otimes \mathbb{1}_{\mathrm{Fil}}(n)$, so that we can restrict our attention to the shifts of the symmetric monoidal unit $\mathbb{1}_{\mathrm{Fil}}$. Now note that the morphisms of spectra $X_n \to X_{n-1}$ induce a map of filtered spectra

$$\begin{tikzcd} X_\star \arrow[d] & \cdots \arrow[r] & X_{n+1} \arrow[r] \arrow[d] & X_n \arrow[r] \arrow[d] & \cdots \\ X(1)_\star & \cdots \arrow[r] & X_n \arrow[r] & X_{n-1} \arrow[r] & \cdots, \end{tikzcd}$$

[6] The notation Déc refers to the *décalage* of a filtered object. A similar construction has been studied starting with filtered spectra and taking Whitehead covers for a t-structure on filtered spectra called the Beilinson t-structure in [Hed20], where its relation to the induced spectral sequences is studied in depth. However, there is no known connection between the décalage presented there (constructed in such a way to recover Deligne's décalage of chain complexes) and the décalage of a cosimplicial object used in this work.

where the vertical morphisms are the same as the horizontal ones. We denote this morphism $X_\star \to X(1)_\star$ by τ. By abuse of notation, τ will usually refer to this morphism on the unit $\mathbb{S}$, but by our previous observation it can be tensored up to any other filtered spectrum.

Example 5.2. Given such an endomorphism on every object, we can single out several classes of filtered spectra.

1. Consider the full subcategory of $\mathrm{Sp}^{\mathrm{Fil}}$ of filtered spectra on which τ acts invertibly. This means that $\tau : X_\star \to X(1)_\star$ must be an equivalence. Using our diagram above, this means that the structure maps $X_{n+1} \to X_n$ must be equivalences for every $n \in \mathbb{Z}$, so that $X_\star$ is essentially a constant filtration. In fact, this means that every τ-invertible filtered spectrum is equivalent to a filtered spectrum of the form $\mathrm{Cs}(X)_\star$. If $\mathrm{Sp}^{\mathrm{Fil}}[\tau^{-1}]$ denotes the full subcategory in question, we conclude that the restriction

$$\mathrm{Cs} : \mathrm{Sp} \to \mathrm{Sp}^{\mathrm{Fil}}[\tau^{-1}]$$

is an equivalence of categories. The inclusion $i : \mathrm{Sp}^{\mathrm{Fil}}[\tau^{-1}] \subset \mathrm{Sp}^{\mathrm{Fil}}$ has an obvious left adjoint given by τ-inversion, i.e. by sending a filtered spectrum $X_\star$ to the colimit

$$\tau^{-1}X_\star = \mathrm{colim}\,(X_\star \xrightarrow{\tau} X(1)_\star \xrightarrow{\tau} X(2)_\star \to \cdots).$$

If we now fix some level $n \in \mathbb{Z}^{\mathrm{op}}$, it is clear that

$$\tau^{-1}X_n = \mathrm{colim}\,(X_n \to X_{n-1} \to X_{n-2} \to \cdots) \simeq \mathrm{Re}(X).$$

Since colimits are computed levelwise, we conclude that $\tau^{-1}X$ is the constant filtered spectrum on $\mathrm{Re}(X)$. If we then recall that Cs induced an equivalence between spectra and constant filtered spectra, we see that the adjunction $\mathrm{Re} \dashv \mathrm{Cs}$ is none other than the adjunction $\tau^{-1} \dashv i$ after identifying $\mathrm{Sp}^{\mathrm{Fil}}[\tau^{-1}]$ with Sp using Cs.

2. Next, one could consider the objects of $\mathrm{Sp}^{\mathrm{Fil}}$ that are τ-complete. To simplify notation in the consideration of τ-completions, we will adopt the notation (for any $k \geq 0$)

$$Y/\tau^k_\star = \mathrm{cofib}(Y(-k)_\star \xrightarrow{\tau^k} Y_\star).$$

In fact, since the Day convolution on $\mathrm{Sp}^{\mathrm{Fil}}$ preserves colimits in each variable, we can exchange the cofibre with the degree shift (the latter being given by tensoring with some shift of $\mathbb{1}_{\mathrm{Fil}}$), and see that $Y/\tau^k_\star \simeq Y_\star \otimes \mathbb{1}_{\mathrm{Fil}}/\tau^k_\star$. The second factor in this tensor product is usually denoted $C\tau^k$. If we let $\mathrm{Sp}^{\mathrm{Fil}\wedge}_{\tau}$ denote the full subcategory on τ-complete filtered spectra, we note that the inclusion $\mathrm{Sp}^{\mathrm{Fil}\wedge}_{\tau} \subset \mathrm{Sp}^{\mathrm{Fil}}$ admits a left adjoint. This left adjoint is given by τ-completion[7]

$$\mathrm{Sp}^{\mathrm{Fil}} \to \mathrm{Sp}^{\mathrm{Fil}\wedge}_{\tau} : Y \mapsto Y^\wedge_\tau = \lim(\cdots \to Y/\tau^2 \to Y/\tau \to Y).$$

Note that this expression is the general expression for completion along an endomorphism in any presentable stable ∞-category, and is not unique to filtered spectra. However, we do make use of this explicit expression for τ-completion to make the following observation. Suppose we are given a

[7] For simplicity in notation, we will sometimes omit the subscript $\star$ that denoted a filtered object. By convention, $\star$ is always indexed on $\mathbb{Z}^{\mathrm{op}}$, while subscripts $*$ and $\bullet$ are indexed on $\mathbb{Z}^{\delta}$ and Δ^{op} respectively.

τ-complete filtered spectrum, w.l.o.g. of the form $Y^\wedge_\tau$, then one can consider its intersection

$$\begin{aligned}\mathrm{In}(Y^\wedge_\tau) &= \lim_{n\in\mathbb{Z}^{\mathrm{op}}} (Y^\wedge_\tau)_n,\\ &\simeq \lim_{n\in\mathbb{Z}^{\mathrm{op}}} \lim(\cdots \to (Y/\tau^3)_n \to (Y/\tau^2)_n \to (Y/\tau)_n),\\ &\simeq \lim_{n\in\mathbb{Z}^{\mathrm{op}}} \lim_{k\geq 1}(Y/\tau^k)_n,\\ &\simeq \lim_{n\in\mathbb{Z}^{\mathrm{op}}} \lim_{k\geq 1} Y_n/Y_{n+k},\\ &\simeq \lim_{n\in\mathbb{Z}^{\mathrm{op}}} Y_n/\mathrm{In}(Y),\\ &\simeq \mathrm{In}(Y)/\mathrm{In}(Y),\\ &\simeq 0.\end{aligned}$$

In this argument, we used that cofibre sequences in the stable ∞-category Sp are equivalent to fibre sequences hence commute with limits. The identification of the n-th part of Y/τ^k comes from the definition of τ^k as the canonical map $Y(-k) \to Y$, so that the cofibres take the form Y_n/Y_{n+k}. We conclude that all τ-complete filtered spectra are such that their intersection vanishes, i.e. what are usually called complete filtrations.

In fact, one can use the formulae above to give a more simple description of τ-completion for a filtered spectrum Y: it is simply given by

$$(Y^\wedge_\tau)_n \simeq Y_n/\mathrm{In}(Y).$$

We conclude that the full subcategories of τ-complete filtered spectra and filtered spectra with vanishing intersection are equivalent.

Further, let us note that τ-complete filtered spectra also admit a description as a right orthogonal complement in the sense of Dwyer–Greenless. Indeed, since there was a tautological adjunction $\mathrm{Cs} \dashv \mathrm{In}$, we see that if Y is a τ-complete filtered spectrum, i.e. such that the spectrum $\mathrm{In}(Y)$ is trivial, then by the Yoneda lemma this is equivalent to

$$\mathrm{map}_{\mathrm{Sp}}(X, \mathrm{In}(Y)).$$

vanishing for all choices of X. Applying the adjunction $\mathrm{Cs} \dashv \mathrm{In}$, we see that this is equivalent to

$$\begin{aligned}0 &\simeq \mathrm{map}_{\mathrm{Sp}}(X, \mathrm{In}(Y)),\\ &\simeq \mathrm{map}_{\mathrm{Sp}^{\mathrm{Fil}}}(\mathrm{Cs}(X), Y).\end{aligned}$$

Now $\mathrm{Cs}(X)$ ranges over all constant filtrations, i.e. all objects of $\mathrm{Sp}^{\mathrm{Fil}}[\tau^{-1}]$, so we see that Y lies in the right orthogonal complement of this subcategory. In the remainder of this text, we will therefore not distinguish between the full subcategories on τ-complete filtered spectra, complete filtrations, and the right orthogonal complement of $\mathrm{Sp}^{\mathrm{Fil}}[\tau^{-1}]$; in fact all three will be denoted $\mathrm{Sp}^{\mathrm{Fil}}{}^\wedge_\tau$.

In fact, having constructed τ, we see that it controls many of the interesting functors in and out of filtered spectra. In particular, let us consider its cofibre when viewed as a map $\mathbb{1}_{\mathrm{Fil}}(-1) \to \mathbb{1}_{\mathrm{Fil}}$. In that case, we have

$$C\tau = \cdots \to 0 \to 0 \to \mathbb{S} \to 0 \to 0 \to \cdots,$$

with the sphere spectrum sitting in filtration degree zero. If we let X be any other filtered spectrum, we see that

$$\begin{aligned}C\tau \otimes X &\simeq \mathrm{cof}(X(-1) \xrightarrow{\tau} X),\\ &\simeq \mathrm{gr}_\star X.\end{aligned}$$

Indeed, in every degree n, this cofibre looks like the cofibre of the structure map $X_{n+1} \to X_n$, which is precisely $\mathrm{gr}_n X$. Now the connecting maps in this filtered spectrum are induced by

$$\mathrm{gr}_n X = X_n/X_{n+1} \to X_{n-1}/X_{n+1} \to X_{n-1}/X_n = \mathrm{gr}_{n-1} X.$$

It is clear that this composite is trivial, i.e. the maps between graded pieces are induced precisely by the maps that are coned out in their construction. We conclude that $\mathrm{gr}_\star X$ should rather be viewed as a graded spectrum, so that

$$C\tau \otimes - : \mathrm{Sp}^{\mathrm{Fil}} \to \mathrm{Sp}^{\mathrm{Gr}}$$

is none other than the associated graded functor gr.

Remark 5.2. This statement is not entirely precise, since we have made an identification between a graded spectrum, and a graded spectrum whose structure maps are all zero. We denote the functor adjoining all of these zero structure maps to obtain a filtered spectrum by

$$P : \mathrm{Sp}^{\mathrm{Gr}} \to \mathrm{Sp}^{\mathrm{Fil}}.$$

We claim that this is actually the right adjoint of the associated graded functor

$$\mathrm{gr} : \mathrm{Sp}^{\mathrm{Fil}} \to \mathrm{Sp}^{\mathrm{Gr}}.$$

Indeed, we saw that this preserved colimits since it is expressed in terms of a collection of cofibres. To prove this, let $X_\star$ be an arbitrary filtered spectrum, and $P(Y)_\star$ the image under P of some graded spectrum Y_*, i.e. such that all of its structure maps are zero. We then simply compute that the mapping space

$$\mathrm{map}_{\mathrm{Sp}^{\mathrm{Fil}}}(X_\star, P(Y)_\star) \subset \prod_{n\in\mathbb{Z}} \mathrm{map}_{\mathrm{Sp}}(X_n, P(Y)_n)$$

in filtered spectra consists of a collection of maps $X_n \to P(Y)_n = Y_n$ such that the obvious squares commute. For every n, this square is of the form

$$\begin{tikzcd} X_{n+1} \arrow[r, "i_n"] \arrow[d, "f_{n+1}"] & X_n \arrow[d, "f_n"] \\ Y_{n+1} \arrow[r, "0"] & Y_n. \end{tikzcd}$$

In particular, this forces the composite $f_n i_n$ to be zero, so that the map f_n factors though the cofibre

$$x_n \to \mathrm{cof}(i_n) =: \mathrm{gr}_n X \to Y_n$$

by the universal property of the latter. We iterate this for every n and come to the conclusion that

$$\mathrm{map}_{\mathrm{Sp}^{\mathrm{Fil}}}(X_\star, P(Y)_\star) \simeq \prod_{n\in\mathbb{Z}} \mathrm{map}_{\mathrm{Sp}}(\mathrm{gr}_n X, Y_n) \simeq \mathrm{map}_{\mathrm{Sp}^{\mathrm{Gr}}}(\mathrm{gr}_* X, Y_*)$$

exhibiting the adjunction $\mathrm{gr} \dashv P$ as required.

In fact, the point of this discussion is primarily to show that P is the right adjoint of a strict symmetric monoidal functor, hence is lax monoidal itself. This means that the object $C\tau$ in filtered spectra, since it can be written as

$$C\tau = P\mathrm{gr}_* 1_{\mathrm{Fil}}$$

admits the structure of an $\mathbb{E}_\infty$-algebra in $\mathrm{Sp}^{\mathrm{Fil}}$, as the image under (lax) monoidal functors of the unit, which is naturally an $\mathbb{E}_\infty$-algebra. Although this observation seems rather innocuous in the context of filtered spectra, equipping $C\tau$ with the structure of an $\mathbb{E}_\infty$-algebra in more general deformed homotopy theories is usually highly nontrivial. The upshot of a theory of deformations of homotopy theories in terms of symmetric monoidal left adjoints from $\mathrm{Sp}^{\mathrm{Fil}}$, as will be extensively discussed below, is then that one can obtain this $\mathbb{E}_\infty$-algebra object formally in any deformation. In fact, once we discuss synthetic spectra, it will become clear that proving that the synthetic version of $C\tau$ is an $\mathbb{E}_\infty$-algebra is highly nontrivial.

5.2 Complete filtrations as cochain complexes

In fact, this last subcategory of τ-complete filtered spectra also admits a different description as an ∞-category of cochain complexes in spectra. The latter picture can be useful for several reasons. In particular, it allows us to consider a new t-structure on τ-complete filtered spectra which arises from the levelwise t-structure on cochain complexes (for brevity, this will not be discussed in this work). Further, this equivalence is symmetric monoidal, where the symmetric monoidal structure on the ∞-category of cochain complexes is given by the more intuitive and familiar notion of the tensor product of cochain complexes.

Definition 5.3 ([Ari21]). Let $\mathcal{C}$ be a stable presentably symmetric monoidal ∞-category, and consider the following indexing category

$$\mathrm{Ch}_0 = \mathbb{Z} \cup \{*\}$$

whose set of objects consists of the integers with a freely adjoined zero object. We then adjoin all morphisms

$$\mathrm{id}_n : n \to n, \qquad \partial_n : n-1 \to n$$

in the poset of integers, as well as making $*$ into the zero object by adding unique maps in and out of it. These are subject to the relation

$$\partial_n \circ \partial_{n-1} = 0$$

as well as the obvious relations concerning the identities. The ∞-category freely generated by these morphisms modulo the relation above is the indexing category Ch of cochain complexes.

Definition 5.4 ([Ari21]). Given a stable presentably symmetric monoidal ∞-category $\mathcal{C}$, we define the ∞-category of cochain complexes in Sp to be the full subcategory

$$\mathcal{K}(\mathcal{C}) = \mathcal{F}\mathrm{un}_*(\mathrm{Ch}, \mathcal{C}) \subset \mathcal{F}\mathrm{un}(\mathrm{Ch}, \mathcal{C})$$

of reduced functors from Ch to $\mathcal{C}$.

It is clear that an object of $\mathcal{K}(\mathcal{C})$ can be interpreted as a cochain complex, since it consists of an increasingly filtered object such that the composite of any structure maps factors through a zero object of $\mathcal{C}$. Given an object X of $\mathcal{K}(\mathcal{C})$, we denote $X(\partial_n)$ by ∂^n. Now using the theory of deformations, in 7.1, we have given an adjunction

$$\lambda : \mathrm{Sp}^{\mathrm{Fil}} \rightleftarrows \mathcal{K}(\mathrm{Sp}) : \rho.$$

Such that

$$\lambda(X) \simeq \Sigma^* \mathrm{gr}_* X, \qquad \rho(C) = \mathrm{Map}(\mathbb{1}_{\mathcal{K}}(\star^{\mathrm{op}}), C),$$

where $\mathbb{1}_{\mathcal{K}}(n)$ is the cochain complex with the spectrum $\mathbb{S}^n$ sitting in degree n and zeroes elsewhere. In this section, we want to show that this restricts to an equivalence

$$\mathrm{Sp}^{\mathrm{Fil}}{}^{\wedge}_{\tau} \simeq \mathcal{K}(\mathrm{Sp}).$$

First, let us show that it restricts, i.e. that there is a factorisation

$$\lambda : \mathrm{Sp}^{\mathrm{Fil}} \xrightarrow{(-)^{\wedge}_{\tau}} \mathrm{Sp}^{\mathrm{Fil}}{}^{\wedge}_{\tau} \to \mathcal{K}(\mathrm{Sp}).$$

In fact, this is obvious. Indeed, by our discussion of Dwyer–Greenlees theory, it is clear that the τ-complete filtrations were precisely the $C\tau$-local objects. By this reasoning, it suffices to show that λ carries $C\tau$-equivalences to equivalences in $\mathcal{K}(\mathrm{Sp})$. However, since tensoring with $C\tau$ simply recovers the associated graded of a filtration, we see that $C\tau$-equivalences are none other than graded equivalences. Now using the explicit formula $\lambda \simeq \Sigma^* \mathrm{gr}_*$, it is clear that this is true. On the other hand, if we want to show that ρ factors through the inclusion of complete filtrations, it suffices to compute the limit of the filtered spectra arising in the image of ρ. Letting C be a cochain complex of spectra, we obtain

$$\begin{aligned}\lim_{n\in\mathbb{Z}^{\mathrm{op}}} \mathrm{Map}(\mathbb{1}_{\mathcal{K}}(n), C) &\simeq \mathrm{Map}(\operatorname*{colim}_{n\in\mathbb{Z}^{\mathrm{op}}} \mathbb{1}_{\mathcal{K}}(n), C),\\ &\simeq \mathrm{Map}(0, C),\\ &\simeq 0\end{aligned}$$

The central observation to make is that this mapping space would encode (along with infinitely many zero maps) a map out of the spectrum obtained as the transfinite iterated pushout of $\mathbb{S}$ along zero objects, and it is clear that the latter is equivalent to

$$\mathbb{S}^\infty \simeq \Sigma^\infty S^\infty \simeq 0,$$

whence the vanishing above follows. We conclude that the adjunction $\lambda \dashv \rho$ restricts as

$$\lambda : \mathrm{Sp}^{\mathrm{Fil}^\wedge_\tau} \rightleftarrows \mathcal{K}(\mathrm{Sp}) : \rho,$$

and use the same notation. Now we want to show that this adjunction is part of an equivalence of ∞-categories. We will roughly follow the argument that is worked out in detail in Section 3 of [Ari21]. For this, let us first note that λ is conservative. Indeed, since all Σ^n are conservative functors, we see that λ is conservative if and only if the family $\{\mathrm{gr}_*\}$ is conservative. This is none other than a restatement of the fact that τ-complete filtered spectra are obtained from filtered spectra by localisation at the $C\tau$-equivalences, the latter being precisely the graded equivalences. This has the upshot that one can utilise the commutative triangles associated to any adjunction to simplify analysis of the unit and co-unit transformations. Indeed, part of the datum of an adjunction is a triangle

$$\begin{tikzcd} \lambda \arrow[r, "\lambda\eta"] \arrow[rr, bend right, "\mathrm{id}_\lambda"'] & \lambda\rho\lambda \arrow[r, "\epsilon"] & \lambda, \end{tikzcd}$$

where η, ϵ denote the unit and counit of $\lambda \dashv \rho$ respectively. In particular, we can use the two-out-of-three rule for equivalences to see that $\lambda\eta$ is an equivalence if ϵ is, and conservativity of λ guarantees that this is equivalent to η being an equivalence. In conclusion, it suffices to show that the counit of the adjunction is a natural equivalence. Given some cochain complex X, we describe $\lambda\rho C$ as

$$\begin{aligned}\lambda\rho C &\simeq \lambda\mathrm{Map}(\mathbb{1}_{\mathcal{K}}(\star^{\mathrm{op}}), C),\\ &\simeq \Sigma^*\mathrm{gr}_*\mathrm{Map}(\mathbb{1}_{\mathcal{K}}(\star^{\mathrm{op}}), C),\\ &\simeq \Sigma^*\mathrm{cof}(\mathrm{Map}(\mathbb{1}_{\mathcal{K}}(*+1), C) \to \mathrm{Map}(\mathbb{1}_{\mathcal{K}}(*), C)),\\ &\simeq \Sigma^*\mathrm{fib}(\Sigma\mathrm{Map}(\mathbb{1}_{\mathcal{K}}(*+1), C)) \to \Sigma\mathrm{Map}(\mathbb{1}_{\mathcal{K}}(*), C)),\\ &\simeq \Sigma^{*+1}\mathrm{Map}(\mathrm{cof}(\mathbb{1}_{\mathcal{K}}(*) \to \mathbb{1}_{\mathcal{K}}(*+1)), C),\\ &\simeq \Sigma^{*+1}\mathrm{Map}(\mathbb{S}^{*+1}, C^*),\\ &\simeq \Sigma^{*+1}\Sigma^{-*-1}\mathrm{Map}(\mathbb{S}, C^*),\\ &\simeq C.\end{aligned}$$

The final step requires an explicit description of the cofibre appearing in the mapping spectrum. This can be observed rather easily by writing out the terms in the cofibre sequence, i.e. (fixing $* = n$):

$$\begin{tikzcd} \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{S}^n \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \mathbb{S}^{n+1} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & 0 \arrow[r] & \mathbb{S}^{n+1} \arrow[r] & \mathbb{S}^{n+1} \arrow[r] & 0 \arrow[r] & \cdots . \end{tikzcd}$$

Indeed, one can compute these cofibres levelwise. Further, it is clear that

$$\mathrm{cof}(\mathbb{S}^n \to 0) \simeq \Sigma\mathrm{fib}(\mathbb{S}^n \to 0) \simeq \Sigma\mathbb{S}^n \simeq \mathbb{S}^{n+1}, \qquad \mathrm{cof}(0 \to \mathbb{S}^{n+1}) \simeq \mathbb{S}^{n+1}.$$

In fact, the connecting map between the two copies of $\mathbb{S}^{n+1}$ appearing in the cofibre is the identity, since the map between the original cochain complexes was precisely the one induced by the identity from the pushout of $\mathbb{S}^n$ along two zero objects. The fact that this connecting map is the identity is salient. Indeed, it means that a map of cochain complexes out of the quotient above amounts to the datum of countably many trivial maps, and two maps from $\mathbb{S}^{n+1}$ to the n-th and $(n+1)$st spectra of the target, but subject to the relation that they are essentially the same map so that all necessary diagrams may commute. We conclude that such a map is equivalent to a single map out of $\mathbb{S}^{n+1}$ to C^n, and the result follows. We conclude that the counit ϵ is an equivalence, whence we have proved that there is an equivalence

$$\mathrm{Sp}^{\mathrm{Fil}^{\wedge}_{\tau}} \simeq \mathcal{K}(\mathrm{Sp}).$$

6 Recollements

In this section, we introduce the notion of recollements of ∞-categories, as adapted for stable and symmetric monoidal ∞-categories in work of Shah [Sha21] and Barwick–Glasman [BG16]. The notion of a recollement arises naturally in algebraic geometry, and was first formalised in its modern form in work of Beĭlinson–Bernstein–Deligne [DBB83]. Geometrically, recollements arise when some base space–be it a scheme, topological space, or stack–is decomposed into an open and closed part, so that inclusions of these two parts induce a variety of morphisms between information on the total space, the closed part, and the open part. This information could be encoded in categories of sheaves, quasicoherent sheaves, perverse sheaves, etc. The abstract deformations we consider in this work, will arise as recollements of quasicoherent sheaves on a certain geometric spectral stack.

We begin with a general description of symmetric monoidal recollements due to Shah in [Sha21], building on a definition by Lurie in [Lur17]. Note that recollements can be defined for any ∞-category with finite limits, but we are primarily interested in the case of stable ∞-categories.

Definition 6.1. The datum of a recollement is a diagram of stable ∞-categories

$$\begin{tikzcd} \mathcal{X}_0 \arrow[r, "i"] & \mathcal{X} \arrow[l, bend right, "i_L"'] \arrow[r, "q"] & \mathcal{X}_1 \arrow[l, bend left, "q_R"] \end{tikzcd}$$

and adjoint functors $i_L \dashv i$, $q \dashv q_R$. These are required to satisfy certain conditions

- The functors i and q_R are fully faithful
- The composite qi is trivial, i.e. constant at the zero object of $\mathcal{Z}$.
- The pair (q, i_L) is jointly conservative, i.e. an arrow f in $\mathcal{X}$ is an equivalence if and only if $q(f)$ and $i_L(f)$ are equivalences.

We call $i_L q_R$ the **gluing functor**, $\mathcal{X}_0$ the **open** part and $\mathcal{X}_1$ the **closed** part of the recollement.

In fact, since these conditions are rather strict, we will see that a recollement can be recovered entirely from its gluing functor. Alternatively, one could think of a recollement as a generalised fracture square, and it turns out that the structure of a recollement can be recovered entirely from its closed part. Both of these perspectives will be quantified later on.

Remark 6.1. There are two opposite conventions for what the open and closed part of a recollement should be, and the literature is divided between the two. The convention used here is consistent with an open-closed decomposition in algebraic geometry, and the ensuing recollement on quasicoherent sheaves. It is the one adopted in [AMR19]. Most other sources such as [Lur17] and [Sha21] use the opposite convention, which arises from an open-closed decomposition of a topological space and the corresponding recollement on constructible sheaves. To facilitate translation between these two pictures, we adopt the notation $\mathcal{X}_0$, $\mathcal{X}_1$ as opposed to the more common $\mathcal{U}, \mathcal{Z}$. The subcategory $\mathcal{X}_0$ plays the role of the open subcategory corresponding to the open subscheme in this work as well as [AMR19], while it is referred to as the closed subcategory in other sources.

Remark 6.2. First, let us note that the diagram in the definition of a recollement can be completed further. In particular, according to Remark A.8.5 in [Lur17], the inclusion functor i admits a right adjoint i_R given by

$$i_R(X) = \mathrm{fib}(X \to q(X)).$$

This relies on the fact that one can recover (the essential image under i of) $\mathcal{X}_0$ as the full subcategory of $\mathcal{X}$ on objects whose image under q is zero. Indeed, one inclusion is trivial by the condition that qi be trivial. Conversely, if $q(X) \simeq 0$ then one can consider the unit map

$$\eta_X : X \mapsto i i_L(X).$$

Obviously $i_L(\eta_X)$ is an equivalence, while $q(\eta_X)$ is the map between $q(X) \simeq 0$ and $(qi)(i_L(X)) \simeq 0$ hence is also an equivalence. We conclude by joint conservativity that η_X is an equivalence, and X lies in the i_L-local subcategory $\mathcal{X}_0$.

It is also shown later on in Proposition A.8.13 of op. cit. that q admits a left adjoint q_L, but we omit the proof since it is not enlightening at the moment.

We conclude that the diagram in Definition 6.1 can be upgraded to a diagram of adjunctions

$$\begin{tikzcd} \mathcal{X}_0 \arrow[r, "i"] & \mathcal{X} \arrow[l, bend right, "i_L"'] \arrow[l, bend left, "i_R"] \arrow[r, "q"] & \mathcal{X}_1 \arrow[l, bend right, "q_L"'] \arrow[l, bend left, "q_R"] \end{tikzcd}$$

This setup is useful, since it allows us to quantify the second statement we made about recollements, namely that they encode fracture squares and are equivalent to the datum of a reflective and coreflective subcategory, with the other part being recovered as its orthogonal complement.

Proposition 6.1. *Given a recollement, the unit transformations associated to the adjunctions $i_L \dashv i$ and $q \dashv q_R$ induce a commutative diagram of endofunctors*

$$\begin{tikzcd} \mathrm{id}_{\mathcal{X}} \arrow[r] \arrow[d] & i i_L \arrow[d] \\ q_R q \arrow[r] & i i_L q_R q \end{tikzcd}$$

which is actually Cartesian.

Proof. This is rather immediate to verify, since it suffices to check this after applying the conservative pair (i_L, q). After applying i_L we find

$$\begin{tikzcd} i_L \arrow[r] \arrow[d] & i_L i i_L \arrow[d] \\ i_L q_R q \arrow[r] & i_L i i_L q_R q. \end{tikzcd}$$

By fully faithfulness of i, the composite $i_L i$ is naturally equivalent (precisely by the counit transformation) to the identity so that this diagram is equivalent to

$$\begin{tikzcd} i_L \arrow[r] \arrow[d] & i_L \arrow[d] \\ i_L q_R q \arrow[r] & i_L q_R q, \end{tikzcd}$$

which is obviously Cartesian, since the top and bottom horizontal arrows are equivalences. If we instead apply q, we obtain the diagram

$$\begin{tikzcd} q \arrow[r] \arrow[d] & q i i_L \arrow[d] \\ q q_R q \arrow[r] & q i i_L q_R q. \end{tikzcd}$$

However, we imposed the axiom that the composite qi be trivial, and that q_R is fully faithful (i.e. qq_R is equivalent to the identity by the counit), so that this diagram is equivalent to

$$\begin{tikzcd} q \arrow[r] \arrow[d] & 0 \arrow[d] \\ q \arrow[r] & 0, \end{tikzcd}$$

which is once again obviously Cartesian. $\square$

Remark 6.3. In particular, this Cartesian square of enfodunctors induces for ever $x \in X$ a Cartesian square

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & ii_L X \arrow[d] \\ q_R q X \arrow[r] & ii_L q_R q X. \end{tikzcd}$$

This gives us some sort of reconstruction result in which X can be recovered precisely from the cospan it is the limit of. However, this cospan contains some redundant information, and we would prefer to reconstruction some $X \in \mathcal{X}$ from just a pair of objects in $\mathcal{X}_0, \mathcal{X}_1$ and some gluing information between them. These objects are provided by $i_L X$, qX respectively, and the gluing information is the map

$$i_L X \to i_L q_R q X.$$

Of course, if one begins with a recollement this morphism simply arises as the counit of the adjunction $q \dashv q_R$, but when one tries to reconstruct a recollement from its open and closed parts, this is in fact the only information needed. One sees that the data above recovers X by applying the fully faithful functor i to obtain the morphism

$$ii_L X \to ii_L q_R q X$$

in the vertical column of the cospan one one hand. On the other hand, one starts with the object qX, applies the fully faithful functor q_R, and then applies the unit transformation of $i_L \dashv i$ to obtain the bottom row

$$q_R q x \to ii_L q_R q X.$$

Proposition 6.2. *Given a recollement of $\mathcal{X}$ into full subcategories $\mathcal{X}_0$ and $\mathcal{X}_1$ using the same notation as above, one can reconstruct $\mathcal{X}$ as the lax limit*

$$\mathcal{X} \simeq \mathcal{X}_1 \times_{i_L q_R, \mathcal{X}_0, \mathrm{ev}_1} \mathcal{X}_0^{\Delta^1}$$

in $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}$. In this equivalence, we identify $X \in \mathcal{X}$ with the object

$$(qX, i_L X \to i_L q_R q X)$$

in the limit.

Proof. We refer to the remark above for a sketch of the proof of this equivalence. As for a rigorous proof within the more general context of stratifications, we refer to [AMR19]. □

The perspective above emphasises the characterisation of recollements as encoding fracture squares, decomposing objects into their images in the closed and open part of the recollement.

Proposition 6.3 ([Lur17], A.8.20)**.** *A stable ∞-category $\mathcal{X}$ admits a reflective and coreflective full subcategory $\mathcal{X}_1$ that is closed under equivalences if and only if it can be written as the recollement of $\mathcal{X}_1$ and its orthogonal complement.*

We leave the proof to op. cit, but remark that the construction of an orthogonal complement gives an explicit model of quotients in $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}$, so that one obtains a cofibre sequence of the form

$$\mathcal{X}_1^{\perp} \leftarrow \mathcal{X} \hookleftarrow \mathcal{X}_1$$

that can be completed to a recollement.

Finally, let us introduce a notion of recollement that interacts well with symmetric monoidal structures, in the sense that the open and closed subcategories both inherit essentially unique symmetric monoidal structures. In fact, as we will see later, this makes the reconstruction associated to a recollement symmetric monoidal as well.

Definition 6.2 ([Sha21] 2.20)**.** A **symmetric monoidal recollement** is a recollement such that the induced localisations of $\mathcal{X}$ are compatible with the symmetric monoidal structure. That is, i_L-equivalences and q-equivalences are closed under tensoring with any object of $\mathcal{X}$.

In that case, it follows immediately from the theory of symmetric monoidal localisations that the subcategories $\mathcal{X}_0$ and $\mathcal{X}_1$ admit essentially unique symmetric monoidal structures such that the composites ii_L and $q_R q$ are strong symmetric monoidal functors.

Remark 6.4. In fact, if we are given a symmetric monoidal recollement of $\mathcal{X}$, then the reconstruction

$$\mathcal{X} \simeq \mathcal{X}_1 \times_{i_L q_R, \mathcal{X}_0, \mathrm{ev}_1} \mathcal{X}_0^{\Delta^1}$$

of Proposition 6.2 is actually an equivalence of symmetric monoidal ∞-categories. The symmetric monoidal structure on the latter is defined as the pullback of the ∞-operads $\mathcal{X}_0^{\otimes}$, $\mathcal{X}_1^{\otimes}$ and $(\mathcal{X}_0^{\otimes})^{\Delta^1}$ and can be defined on objects by

$$(x_1, \alpha : x_0 \to i_L q_R x_1) \otimes (x_1', \alpha' : x_0 \to i_L q_R x_1') = (x_1 \otimes x_1', x_0 \otimes x_0' \xrightarrow{\alpha\otimes\alpha'} i_L q_R x_0 \otimes i_L q_R x_0' \to i_L q_R (x_0 \otimes x_0')),$$

where the final arrow in the composite uses the assumption that our recollement was symmetric monoidal to deduce that the gluing functor $i_L q_R$ is lax monoidal.

Example 6.1. Let $\mathcal{C}$ be an element of $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}})$ with a dualisable commutative algebra object A. Recall that we could identify full subcategories of $\mathcal{C}$ on A-invertible, A-complete, and A-torsion objects, the latter two being equivalent. These sit inside the diagram

$$\begin{tikzcd} \mathcal{C}[A^{-1}] \arrow[r, hook] & \mathcal{C} \arrow[l, bend right=60, "A^{-1}"'] \arrow[l, bend left=60] \arrow[r, "(-)^{\wedge}_A"] & \mathcal{C}^{\wedge}_A \arrow[l, hook, bend right=60] \arrow[l, hook, bend left=60] \end{tikzcd}$$

of adjoint functors. We claim that this forms a recollement. For this, one would simply have to check that the A-completion of an A-invertible object vanishes, and that the A-completion with the A-inversion functor form a conservative pair.

Proof. The first statement is immediate: given our explicit formula of the A-completion functor, since we see that any A-invertible object vanishes after tensoring with A. Indeed, this is Proposition 3.11 in [MNN17], and just follows from dualisability of A, allowing us to express a mapping space into some $X \otimes A$ with X A-invertible as a mapping space between A-torsion and A-invertible objects, which vanishes by the definition of the latter. The conclusion follows from the Yoneda lemma. In the language of recollements, we have verified that $qi = 0$.

For the second statement, suppose that $f : X \to Y$ is a morphism in $\mathcal{C}$ such that both of the maps

$$f[A^{-1}] : X[A^{-1}] \to Y[A^{-1}], \qquad f^{\wedge}_A : X^{\wedge}_A \to Y^{\wedge}_A$$

are equivalences. We want to show that f is an equivalence. First, recall that the A-complete and A-torsion subcategories were equivalent. This means that we can replace $f^{\wedge}_A$ with its (inverse) image under this equivalence, namely its image under the right adjoint $\mathcal{C} \to {}_A\mathcal{C}$ given by $A \otimes -$. Therefore, we have simplified to the assumption that $A \otimes f$ and $f[A^{-1}]$ are equivalences.

Now showing that f is an equivalence is tautologically equivalent to showing that $\mathbb{1}_{\mathcal{C}} \otimes f$ is an equivalence, where $\mathbb{1}_{\mathcal{C}}$ is the monoidal unit of $\mathcal{C}$. Therefore, if we let $\mathcal{E}$ denote the subcategory of $\mathcal{C}$ on objects Z such that $Z \otimes f$ is an equivalence, we want to show that it contains the unit. It is clear that this subcategory $\mathcal{E}$ is nonempty, as it contains A, an dis closed under colimits, since the tensor product commutes with these. We now hand over the proof to [MNN17], where a simple argument is presented as to how the unit sits in the middle of a fibre sequence between objects that lie in $\mathcal{E}$ either assumption, or by a simple argument. The construction of this fibre sequence would go beyond the scope of this section, as it requires an identification of the Adams filtration.

We conclude that $\{q, i_L\}$ form a conservative pair $\square$

Bringing it all together, we see that any stable presentably symmetric monoidal ∞-category $\mathcal{C}$ with a dualisable homotopy associative algebra object A gives rise to a recollement of $\mathcal{C}$ in terms of $\mathcal{C}[A^{-1}]$ and $\mathcal{C}^{\wedge}_{A}$, where the latter is equivalent to the full subcategory on A-torsion objects. In fact, since A-invertible objects are those that are killed by tensoring with A ([MNN17] Proposition 3.11), we see that the family of $(-)[A^{-1}]$-equivalences is closed under taking tensor products with any object. The same is true for the $(-)^{\wedge}_{A}$-equivalence, since these are defined to be the maps that become equivalence after tensoring with A. We conclude that this recollement is actually a symmetric monoidal recollement.

Remark 6.5. In this example, the gluing functor is given by

$$X \to X^{\wedge}_{A}[A^{-1}],$$

and it sits in the familiar fracture square

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X[A^{-1}] \arrow[d] \\ X^{\wedge}_{A} \arrow[r] & X^{\wedge}_{A}[A^{-1}]. \end{tikzcd}$$

6.1 Filtered spectra as a recollement

In this section, we give the main example of a recollement that is of interest to us. It is given by the structure of a recollement on the ∞-category of filtered spectra, where the open and closed parts correspond to constant and complete filtations respectively. This recollement admits a variety of descriptions: as an $[0 \leq 1]$-stratification of a spectral stack, as a Dwyer–Greenlees type recollement associated to a certain algebra, or also more ad hoc as a closer analysis of the information contained in a general filtered spectrum. The geometric picture in terms of stratifications will be discussed in Section 7.4, and tied into the Dwyer–Greenlees type interpretation to be discussed below. We begin simply with an ad hoc construction of the fundamental recollement.

$$\begin{tikzcd} \mathrm{Sp} \arrow[r, "\mathrm{Cs}"] & \mathrm{Sp}^{\mathrm{Fil}} \arrow[l, bend right, "\mathrm{Re}"'] \arrow[l, bend left, "\mathrm{In}"] \arrow[r, "q"] & \mathrm{Sp}^{\mathrm{Fil}\wedge}_{\tau} \arrow[l, bend right, "q_L"'] \arrow[l, bend left, "q_R"] \end{tikzcd}$$

In this diagram, Cs and its adjoint are well understood as the functors sending a given spectrum to its asociated constant filered spectrum, along with its colimit and limit. The functor q_R is the inclusion of complete filtrations into general filtered spectra, and q is then the previously constructed left adjoint that completes a filtration, i.e. quotients out by the limit in every degree. This admits a further left adjoint q_L. It is clear from this description that both sides of the diagram can be viewed as full subcategories. We claim that they actually constitute a recollement.

Proposition 6.4. *The diagram above endows* $\mathrm{Sp}^{\mathrm{Fil}}$ *with the structure of a recollement with open part* Sp *and closed part* $\mathrm{Sp}^{\mathrm{Fil}\wedge}_{\tau}$.

Proof. Clearly, it suffices to show the last two points in the definition of a recollement; i.e. that the composite $q \circ \mathrm{Cs}$ is constant, and that $\{q, \mathrm{Re}\}$ is a conservative family.

For the first, let us consider an arbitrary spectrum X, then Cs sends this to the constant filtration

$$X \mapsto (\cdots \to X \to X \to X \to \cdots)$$

with identity maps. We have seen that q can be described as quotienting out by the limit, but it is clear that

$$\begin{aligned} \mathrm{In}(\mathrm{Cs}(X)_{\star}) &\simeq \lim_{n \in \mathbb{Z}^{\mathrm{op}}} X, \\ &\simeq X, \end{aligned}$$

so that $X_\star/\mathrm{In}(X)$ is trivial, consisting of $X/X \simeq 0$ in every filtration degree. Finally, let us prove that $\{q, \mathrm{Re}\}$ is a conservative pair. Let

$$f : X_\star \to Y_\star$$

be a morphism of filtered spectra such that

$$\mathrm{Re}(f) : X_{-\infty} \xrightarrow{\sim} Y_{-\infty}, \qquad q(f) : X_\star/X_{+\infty} \xrightarrow{\sim} Y_\star/Y_{+\infty}$$

are equivalences. Note that equivalences in the functor ∞-category $\mathrm{Sp}^{\mathrm{Fil}}$ (hence also its full subcategory of complete filtrations) are detected levelwise, so that the second statement means that for every $n \in \mathbb{Z}$, f induces an equivalence

$$X_n/X_{+\infty} \xrightarrow{\sim} Y_n/Y_{+\infty}.$$

Further, by functoriality of the colimit, f also induces an equivalence

$$X_{-\infty}/X_{+\infty} \xrightarrow{\sim} Y_{-\infty}/Y_{+\infty},$$

where the map $X_{+\infty} \to X_{-\infty}$ whose cofibre we are considering can be viewed as the colimit-to-limit comparison map. Arranging these equivalences and associated cofibre sequences next to each other, we obtain a commutative diagram

$$\begin{tikzcd}
X_{+\infty} \arrow[r] \arrow[d] & X_{-\infty} \arrow[r] \arrow[d] & X_{-\infty}/X_{+\infty} \arrow[d] \\
Y_{+\infty} \arrow[r] & Y_{-\infty} \arrow[r] & Y_{-\infty}/Y_{+\infty}.
\end{tikzcd}$$

All horizontal sequences are cofibre sequences, while the two rightmost vertical maps are equivalences either by assumption on $\mathrm{Re}(f)$, or by the observation above. Now this means that the final leftmost vertical map is an equivalence as well. Indeed, it is a standard fact of stable homotopy theory that given a diagram of cofibre sequences with two out of three vertical maps being equivalences, that the last map is an equivalence as well. One can observe this by noting that applying homotopy groups to this diagram would result in a pair of long exact sequences of abelian groups associated to the cofibrations. In this diagram of long exact sequences, pairs of equivalences alternate with maps that are not necessarily equivalences. However, each of the latter is surrounded by two equivalences, so that one can apply the five lemma from homological algebra to see that it must be an equivalence as well. We conclude that the induced map $\mathrm{In}(f)$ is an equivalence.

Finally, we simply need to recreate the argument above for finite $n \in \mathbb{Z}$, considering the diagram of cofibre sequences of the form

$$\begin{tikzcd}
X_{+\infty} \arrow[r] \arrow[d] & X_n \arrow[r] \arrow[d] & X_n/X_{+\infty} \arrow[d] \\
Y_{+\infty} \arrow[r] & Y_n \arrow[r] & Y_n/Y_{+\infty}.
\end{tikzcd}$$

We see that the leftmost vertical map is an equivalence by the preceding paragraph, while the rightmost vertical map is an equivalence by the assumption on $q(f)$. We conclude by the same incarnation of the five lemma that the central map

$$f : X_n \to Y_n$$

is an equivalence. We conclude that f was an equivalence. □

Remark 6.6. Note that the five lemma for cofibre sequences of spectra is agnostic with regards to which of the two maps in a diagram of cofibre sequences are equivalences. In fact, one sees that the argument for why $\mathrm{In}(f)$ is an equivalence when $q(f)$ and $\mathrm{Re}(f)$ are, equally tells us that $\mathrm{Re}(f)$ is an equivalence whenever $\mathrm{In}(f)$ and $q(f)$ are. Therefore, we may equivalently view $\{q, \mathrm{In}\}$ as a conservative pair, skipping straight to the final argument of the proof above. This conservative pair may feel more intuitive, since it essentially tells us that a filtration decomposes into its limit and the filtration modulo the limit. In fact, using Section 5.2, one can describe this recollement as

$$\begin{tikzcd} \mathrm{Sp} \arrow[r, "\mathrm{Cs}"] & \mathrm{Sp}^{\mathrm{Fil}} \arrow[l, bend right, "\mathrm{Re}"'] \arrow[l, bend left, "\mathrm{In}"] \arrow[r, "\lambda"] & \mathcal{K}(\mathrm{Sp}). \arrow[l, bend right, "q_L"'] \arrow[l, bend left, "\rho"] \end{tikzcd}$$

Under this equivalence, our conservative pairs $\{q, \mathrm{Re}\}$ and $\{q, \mathrm{In}\}$ transform into the conservative pairs $\{\lambda = \Sigma^* \mathrm{gr}_*, \mathrm{Re}\}$, $\{\Sigma^* \mathrm{gr}_*, \mathrm{In}\}$. Now suspensions are conservative, so we can rephrase this as the existence of a conservative pair

$$\{\mathrm{gr}_*, \mathrm{In}\}.$$

The latter conservative pair is common in most literature about filtered spectra, e.g. Proposition II.1.9 in [Hed20] based on Remark 2.16 in [GP18], and applied in [BMS18], where the existence of this conservative pair appears in Lemma 5.2.

Example 6.2. Apart from the *ad hoc* construction of a recollement on filtered spectra given above, we can also express this in the form of Example 6.1. Indeed, in Section 5.1, we identified the full subcategory of constant filtrations with the τ -invertible filtered spectra, while the τ-complete objects were identified with complete filtrations. This means that if we set up a recollement based on the dualisable commutative algebra object $C\tau$ of the form

$$\begin{tikzcd} \mathrm{Sp}^{\mathrm{Fil}}[\tau^{-1}] \arrow[r] & \mathrm{Sp}^{\mathrm{Fil}} \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathrm{Sp}^{\mathrm{Fil}}{}^{\wedge}_{\tau}, \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}$$

then this is none other than the recollement we constructed earlier. In fact, we can give filtered spectra a third characterisation in terms of spectral algebraic geometry, and the recollement associated to this characterisation will also be the same as the one constructed above. This appears in Proposition 7.6.

7 Deformations

We now come to the main theoretical chapter of this work, which is the discussion of deformations of homotopy theories. This section is divided into describing two models of deformations of homotopy theories. One is based on spectral algebraic geometry, inspired by [Mou21], while the second model has appeared in [BHS20]. We show that these two models agree, and interpolate between them to work out certain examples and describe certain recollements arising canonically on deformations of stable homotopy theory. Finally, in the next sections on synthetic spectra, we consider the primary example of such a deformation and see the machinery from this section in action.

7.1 Geometric deformations

In this section, we will introduce the notion of a deformation of homotopy theories from the geometric standpoint. Indeed, one-parameter deformations can be interpreted geometrically as objects living over $\mathbb{A}^1$ such that all fibres over nonzero points of $\mathbb{A}^1$ are canonically equivalent to some generic fibre, while the deformation degenerates into the special fibre over 0. For us, this geometric picture needs to be converted to a statement about homotopy theories, so that we convert the geometric intuition about deformations into a statement about presentable stable ∞-categories.

In fact, this process is part of the theory of *noncommutative geometry*, in which geometric objects such as (spectral) stacks are replaced by their derived (∞-)categories of quasicoherent sheaves. This has been done in the ∞-categorical setting in e.g. [AMR19], where presentable stable ∞-categories are viewed as noncommutative stacks. In fact, op. cit. develops a theory of stratifications for these noncommutative stacks which in particular recovers the theory of recollements. In the latter, the words *noncommutative stack* and *stable homotopy theory* both refer to an object of $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}})$.

7.1.1 The geometric stack $\mathbb{A}^1/\mathbb{G}_m$

As described above, a geometric one-parameter deformation with a single special fibre can be described as a family over $\mathbb{A}^1$, with all fibres over $\mathbb{A}^1 \setminus 0$ canonically equivalent. In fact, one can view the equivalences between all nonzero fibres as being induced by the action of $\mathbb{G}_m$ on the latter, so that our deformations of interest could equivalently be described as objects over the quotient stack $\mathbb{A}^1/\mathbb{G}_m$. In this section, we will review work of [Mou21] on the computation of the noncommutative stack associated to $\mathbb{A}^1/\mathbb{G}_m$. The main result is the equivalence

$$\mathrm{QCoh}(\mathbb{A}^1/\mathbb{G}_m) \simeq \mathrm{Sp}^{\mathrm{Fil}}$$

of presentable stable ∞-categories, which is furthermore symmetric monoidal. The proof of this equivalence requires some work in spectral algebraic geometry, which goes beyond the scope of this paper, hence is primarily relegated to op. cit. Finally, we note that the computation in op. cit. is of independent interest, since it gives a geometric way of constructing filtrations on ∞-categories of quasicoherent sheaves over stacks, and the latter are of particular computational interest for the spectral sequences they provide. In fact, there is a classical (1-categorical) analogue of this result in work of Simpson [Sim90] concerning a similar equivalence with the category of filtered vector spaces.

In spectral algebraic geometry, there are two different notions of the affine line that only coincide rationally. To recover the main result of this section concerning quasicoherent sheaves on a quotient of the affine line, we use what is called the flat affine line in spectral algebraic geometry. This is the affine spectral scheme associated to the $\mathbb{E}_\infty$-ring spectrum obtained as the suspension spectrum of the monoid $\mathbb{N}$ in spaces:

$$\mathbb{A}^1 = \mathbb{A}^1_\flat := \mathrm{Spec}(\mathbb{S}[\mathbb{N}]),$$

where the latter equivalent notations for the suspension spectrum simply emphasises that the latter is to be viewed as a polynomial $\mathbb{E}_\infty$-ring spectrum on one variable. The essential subtlety in spectral algebraic geometry is that this polynomial $\mathbb{E}_\infty$-ring is not the free $\mathbb{E}_\infty$-ring in one variable over $\mathbb{S}$. The latter is denoted $\mathbb{S}\{t\}$ (its spectrum is called the smooth affine line), and obtained as

$$\mathbb{S}\{t\} \simeq \bigoplus_{n\geq 0} \mathbb{S}_{\Sigma_n}.$$

In fact, the suspension spectrum

$$\mathbb{S}[\mathbb{N}] = \mathbb{S}[t] \simeq \bigoplus_{n\geq 0} \mathbb{S}$$

is only the free $\mathbb{E}_1$-algebra on a single variable, but can be upgraded to an $\mathbb{E}_\infty$-algebra. Working with the flat affine line is often preferable, since its underlying ring spectrum is a suspension spectrum, whence it is easy to define maps out of it. Further, the homotopy groups of the latter are easily seen to be given by

$$\pi_*\mathbb{S}[t] \cong (\pi_*\mathbb{S})[t],$$

which is not true for the smooth affine line.

One can then follow a similar construction to obtain the spectral analogue of the multiplicative group scheme, namely:

$$\mathbb{G}_m := \mathrm{Spec}(\mathbb{S}[\mathbb{Z}]),$$

where we now take the suspension spectrum on the discrete (grouplike) monoid $\mathbb{Z}$ in spaces. No notion of an affine line is complete without a $\mathbb{G}_m$-action by scaling, and as in the classical case, this action is obtained as dual to a coaction

$$\mathbb{S}[\mathbb{N}] \to \mathbb{S}[\mathbb{N}] \otimes \mathbb{S}[\mathbb{Z}].$$

The latter arises as the suspension of the map of monoids

$$\mathbb{N} \to \mathbb{N} \times \mathbb{Z}$$

obtained as the product of the identity of $\mathbb{N}$ and the monoidal inclusion $\mathbb{N} \to \mathbb{Z}$. On affine schemes, this then induces the scaling action map

$$\mathbb{G}_m \times \mathbb{A}^1 \to \mathbb{A}^1.$$

Although the stack $\mathbb{A}^1/\mathbb{G}_m$ is not a quotient by a free action, hence should really be interpreted as a higher stack, the quotient is constructed as the realisation of a simplicial object in the ∞-category $\mathcal{S}\mathrm{hv}_{\mathrm{fpqc}}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$ with particularly nice properties, making it into a geometric stack (cf. Corollary 9.3.1.4 in [Lur18]). The ∞-category of quasicoherent sheaves on these stacks are well behaved, since it can be obtained as the totalisation of a cosimplicial diagram of module categories. We list three important properties.

Proposition 7.1 ([Mou21] Proposition 2.2). *Let $\mathcal{X}$ be a geometric stack, then the ∞-category $\mathrm{QCoh}(\mathcal{X})$ is presentably symmetric monoidal, and possesses a natural bicomplete t-structure compatible with the symmetric monoidal structure.*

To further analyse this stack, let us note that $\mathbb{G}_m$ acts freely on $\mathbb{G}_m \subset \mathbb{A}^1$, while it acts trivially on the point $0 \subset \mathbb{A}^1$. Therefore, the quotient can be seen as a point $\mathbb{G}_m/\mathbb{G}_m \simeq \mathrm{Spec}(\mathbb{S})$ and a classifying stack $0/\mathbb{G}_m \simeq B\mathbb{G}_m$. Therefore, we can reconstruct quasicoherent sheaves on the entire quotient stack by gluing together sheaves on $B\mathbb{G}_m$ and $\mathrm{Spec}(\mathbb{S})$. The latter is affine, so that there is a tautological equivalence

$$\mathrm{QCoh}(\mathrm{Spec}(\mathbb{S})) \simeq \mathrm{Mod}(\mathrm{Sp}; \mathbb{S}) \simeq \mathrm{Sp}$$

of stable presentably symmetric monoidal ∞-categories. As for $B\mathbb{G}_m$, let us recall that classically a $\mathbb{G}_m$-action on a scheme induces a grading on quasicoherent sheaves. The same turns out to be true in our situation, and we obtain

$$\mathrm{QCoh}(B\mathbb{G}_m) \simeq \mathrm{Sp}^{\mathrm{Gr}}.$$

Proposition 7.2 ([Mou21], Theorem 4.1). *There is a symmetric monoidal equivalence*

$$\mathrm{QCoh}(B\mathbb{G}_m) \simeq \mathrm{Sp}^{\mathrm{Gr}}$$

between quasicoherent sheaves on the geometric stack $B\mathbb{G}_m$ and graded spectra.

The complete proof requires more spectral algebraic geometry than can be covered here, but we will sketch an outline of the proof for completeness. For a rigorous discussion, see [Mou21].

Proof. Recall that quotient stacks are constructed using a geometric realisation

$$B\mathbb{G}_m \simeq |\mathbb{G}_m^\bullet|,$$

as in [Lur18] Example 9.1.1.7. In particular, this means that the (symmetric monoidal) ∞-category of quasicoherent sheaves on the former can be recovered as a totalisation of a cosimplicial diagram of symmetric monoidal left adjoints obtained as pullbacks along the simplicial structure maps above, following [Lur18] Definition 6.2.2.1. Since $\mathbb{G}_m$ was constructed as $\mathbb{G}_m = \mathrm{Spec}(\mathbb{S}[\mathbb{Z}])$, we see that this fits in an augmented cosimplicial limit diagram in $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}})$, or equivalently in $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}$:

$$\mathrm{QCoh}(B\mathbb{G}_m) \to \mathrm{Mod}(\mathrm{Sp}; \mathbb{S}[\mathbb{Z}]^{\wedge\bullet}).$$

It is therefore sufficient to show that $\mathrm{Sp}^{\mathrm{Gr}}$ also shows up as the limit of this diagram. Indeed, let us choose a new augmentation given by

$$\mathrm{Sp}^{\mathrm{Gr}} \to \mathrm{Mod}(\mathrm{Sp}; \mathbb{S}[\mathbb{Z}]^{\wedge 0}) \simeq \mathrm{Sp} : X_\star \mapsto \bigoplus_{n\in\mathbb{Z}} X_n.$$

This is the left adjoint to the constant functor $\mathrm{Sp} \to \mathrm{Sp}^{\mathrm{Gr}}$, or equivalently the left adjoint to the pullback along the map $\mathbb{Z}^\delta \to \Delta^0$. Therefore, we see that it is a symmetric monoidal left adjoint by the universal property of the Day convolution[8]. To show that the augmented cosimplicial diagram obtained as such is also a limit diagram, we simply need to verify a descent condition formulated in [Lur17] Corollary 4.7.5.3. These conditions are largely formal. Indeed, the augmentation is easily seen to be conservative, since we are working in the additive setting so that any term X_n of a graded spectrum X_* is a retract of its image under the augmentation by the identity composite

$$X_n \to \bigoplus_{m\in\mathbb{Z}} X_m \xrightarrow{\sim} \prod_{m\in\mathbb{Z}} X_m \to X_n.$$

The adjointability conditions holds for any geometric stack, and the condition that $\mathrm{Sp}^{\mathrm{Gr}}$ admits geometric realisations, which are preserved by the augmentation if they are split by it, is an immediate consequence of the fact that $\mathrm{Sp}^{\mathrm{Gr}}$ is equivalent to a $\mathbb{Z}$-indexed product of Sp, so that this can be checked on every factor. □

Using this identification, one can proceed to describe $\mathrm{QCoh}(\mathbb{A}^1/\mathbb{G}_m)$. Once again, we will sketch the proof of this result, which takes up Section 5 in [Mou21]. The key is to consider the essentially unique morphism

$$\widetilde{\phi} : \mathbb{A}^1 \to \mathrm{Spec}(\mathbb{S}),$$

along with its image after quotienting out by the $\mathbb{G}_m$-action on both sides:

$$\phi : \mathbb{A}^1/\mathbb{G}_m \to \mathrm{Spec}(\mathbb{S})/\mathbb{G}_m \simeq B\mathbb{G}_m.$$

This induces an adjunction

$$\phi^* : \mathrm{Sp}^{\mathrm{Gr}} \simeq \mathrm{QCoh}(B\mathbb{G}_m) \rightleftarrows \mathrm{QCoh}(\mathbb{A}^1/\mathbb{G}_m) : \phi_*.$$

In fact, one can show that the morphism ϕ is of a particular type (quasi-affine quasi-representable morphism of sufficiently nice stacks) that guarantees that this adjunction is monadic, i.e. one can identify

$$\mathrm{QCoh}(\mathbb{A}^1/\mathbb{G}_m) \simeq \mathrm{Mod}(\mathrm{QCoh}(B\mathbb{G}_m); \phi_*\mathbb{1}_{\mathrm{QCoh}(\mathbb{A}^1/\mathbb{G}_m)}).$$

[8]This observation is actually a nontrivial consequence of the monoidal properties of ∞-categories of presheaves of spectra on a space viewed as a symmetric monoidal ∞-groupoid, e.g. $\mathbb{Z}^\delta$ or Δ^0. In fact, one can view the result as a variant of parametrised spectra with Day convolution. The claim above is a direct consequence of [ABG18] Proposition 6.12

The algebra object on the right hand side is precisely the pushforward along ϕ of the structure sheaf $\mathcal{O}_{\mathbb{A}^1/\mathbb{G}_m}$. This can be analysed as a spectrum. Indeed, there is a commutative diagram

$$\begin{tikzcd} \mathbb{A}^1 \arrow[r, "\widetilde{\pi}"] \arrow[d, "\widetilde{\phi}"] & \mathbb{A}^1/\mathbb{G}_m \arrow[d, "\phi"] \\ \mathrm{Spec}(\mathbb{S}) \arrow[r, "\pi"] & B\mathbb{G}_m \end{tikzcd}$$

per construction of $\widetilde{\phi}$ and the quotient maps $\widetilde{\pi}, \pi$. This is clearly a pullback square of stacks, so that one may use another property of quasi-affine quasi-representable morphisms as in [Lur18] Proposition 6.3.4.1, which tells us that the induced diagram of left adjoints on QCoh is right adjointable, i.e. such that the Beck–Chevalley transformation

$$\pi^*\phi_* \to \widetilde{\phi}_*\widetilde{\pi}^*$$

is an equivalence. In particular, this tells us that there is an equivalence

$$\begin{aligned} \pi^*\phi_*\mathcal{O}_{\mathbb{A}^1/\mathbb{G}_m} &\simeq \widetilde{\phi}_*\widetilde{\pi}^*\mathcal{O}_{\mathbb{A}^1/\mathbb{G}_m}, \\ &\simeq \widetilde{\phi}_*\mathcal{O}_{\mathbb{A}^1}, \\ &\simeq \mathbb{S}[\mathbb{N}], \end{aligned}$$

where we used that $\widetilde{\pi}^*$ is monoidal, and the fact that the global sections functor $\widetilde{\phi}_*$ takes the underlying spectrum of an $\mathbb{S}[\mathbb{N}]$-module viewed as a quasicoherent sheaf on $\mathbb{A}^1$.

At this stage, we know what the underlying spectrum of $\phi_*\mathcal{O}_{\mathbb{A}^1/\mathbb{G}_m}$ looks like, but not its structure as a graded spectrum. This requires an alternative description of graded spectra as spectra with a comodule structure over the Hopf algebroid $\mathbb{S}[\mathbb{Z}]$, which we will not describe in this summary. In fact, if one analyses the $\mathbb{G}_m$-action on $\mathbb{A}^1$ more closely, which amounts to analysing the $\mathbb{S}[\mathbb{Z}]$-comodule structure on $\mathbb{S}[\mathbb{N}]$ above, one sees ([Mou21] Proposition 5.1) that $\phi_*\mathcal{O}_{\mathbb{A}^1/\mathbb{G}_m}$ is the graded spectrum denoted $\mathbb{S}[t]$ which consists of a sphere spectrum $\mathbb{S}$ in every grading degree $n \leq 0$ and zeroes elsewhere. We conclude that there is an equivalence

$$\mathrm{QCoh}(\mathbb{A}^1/\mathbb{G}_m) \simeq \mathrm{Mod}(\mathrm{QCoh}(B\mathbb{G}_m); \phi_*\mathcal{O}_{\mathbb{A}^1/\mathbb{G}_m}) \simeq \mathrm{Mod}(\mathrm{Sp}^{\mathrm{Gr}}; \mathbb{S}[t]).$$

To identify the right hand side of this chain of equivalences, we see that it fits in the adjunction

$$L : \mathrm{Sp}^{\mathrm{Gr}} \rightleftarrows \mathrm{Sp}^{\mathrm{Fil}} : U,$$

defined in Definition 5.2. Indeed, it is clear that

$$\mathbb{S}[t] \simeq U\mathbb{1}_{\mathrm{Fil}},$$

so that monoidality of the adjunction above (it is induced by a strict monoidal functor $\mathbb{Z}^\delta \to \mathbb{Z}$) gives rise to a factorisation

$$L : \mathrm{Mod}(\mathrm{Sp}^{\mathrm{Gr}}; \mathbb{S}[t]) \rightleftarrows \mathrm{Sp}^{\mathrm{Fil}} : U,$$

which we claim is an equivalence. Of course, this is nothing else than checking that the original adjunction $L \dashv U$ is monadic. This follows immediately from the observation that the adjunction above witnesses $\mathrm{Sp}^{\mathrm{Fil}}$ as an $\mathrm{Sp}^{\mathrm{Gr}}$-algebra in $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}$, so that we can apply our variant of the Schwede–Shipley theorem with $K = \mathbb{Z}^\delta$. Indeed, we see that $\mathrm{Map}^{\mathbb{Z}^\delta}(\mathbb{1}_{\mathrm{Fil}}, -)$ is a conservative functor on $\mathrm{Sp}^{\mathrm{Fil}}$, since equivalences of filtered spectra are precisely maps f of filtered spectra that are equivalences at every level, i.e. such that for all integers n, the map $\mathrm{Map}(\mathbb{1}_{\mathrm{Fil}}(n), f)$ is an equivalence. These assemble for all n to form the equivalences of graded spectra $\mathrm{Map}^{\mathbb{Z}^\delta}(\mathbb{1}_{\mathrm{Fil}}, f)$. Now note that the original adjunction was monoidal, so that the algebra showing up in the Schwede–Shipley result is none other than the image of the unit of $\mathrm{Sp}^{\mathrm{Fil}}$ under the lax monoidal right adjoint, i.e $\mathbb{S}[t]$.

We conclude that the chain of equivalences above can be extended to a symmetric monoidal equivalence

$$\mathrm{QCoh}(\mathbb{A}^1/\mathbb{G}_m) \simeq \mathrm{Mod}(\mathrm{QCoh}(B\mathbb{G}_m); \phi_*\mathcal{O}_{\mathbb{A}^1/\mathbb{G}_m}) \simeq \mathrm{Mod}(\mathrm{Sp}^{\mathrm{Gr}}; \mathbb{S}[\tau]) \simeq \mathrm{Sp}^{\mathrm{Fil}}.$$

Remark 7.1. Unfortunately, the deformation picture given above is not as geometric as one might hope. Indeed, the flat affine line is not naturally a free $\mathbb{E}_\infty$-algebra, so that it induces a variety of complications with respect to $\mathbb{E}_n$-structures. Concretely, if one works in nonconnective spectral algebraic geometry using e.g. even ring spectra as opposed to connective ring spectra, one would have to work with grading shifts of this polynomial $\mathbb{E}_\infty$-algebra, such as even versions, which can no longer be promoted to $\mathbb{E}_\infty$-ring spectra. Fortunately, this problem can be patched on the level of noncommutative stacks. More specifically, the module ∞-category $\mathrm{Mod}(\mathrm{Sp}^{\mathrm{Gr}};\mathbb{S}[t])$ does not depend on any shifts of $\mathbb{S}[t]$. Therefore, we have decided to keep working with the flat affine line, since:

- It is similar in construction to the classical affine line, whose quotient $\mathbb{A}^1/\mathbb{G}_m$ classifies extended Cartier divisors. Furthermore, it has been related to filtrations on cohomology in work of Simpson as previously mentioned.
- It still gives us the correct results on noncommutative stacks, as well as the fundamental intuition behind geometric deformations:
$$\mathrm{QCoh}(\mathbb{A}^1/\mathbb{G}_m) \simeq \mathrm{Sp}^{\mathrm{Fil}}.$$

If we are actually interested in the spectral stacks underlying these deformations, it is important that we use the correct spectral analogue of the classifying stack of extended Cartier divisors–the latter controlling one-parameter deformations of stacks–which is given by the stack $\mathrm{CDiv}^{\dagger}_{\mathrm{eff}}$ of [Gre21]. See section 2.2 in op. cit. for a detailed description of this stacks, as well as the comments brought up in this remark.

7.1.2 The fibres of $\mathbb{A}^1/\mathbb{G}_m$.

To study and define deformations in general, let us begin by analysing the universal deformation, namely the identity map of $\mathbb{A}^1/\mathbb{G}_m$, or equivalently the initial $\mathrm{Sp}^{\mathrm{Fil}}$-algebra: $\mathrm{Sp}^{\mathrm{Fil}}$ itself. Geometrically, we noted that the special fibre at $\tau = 0$ of this deformation is precisely the fibre over $B\mathbb{G}_m$, while the generic fibre consisted of the generic points that get collapsed to $\mathrm{Spec}(\mathbb{S})$. This gives us an immediate definition of what the generic and special fibres of a deformation should be in terms of geometric intuition. Indeed, note that there are obvious inclusion morphisms of spectral stacks

$$\mathrm{Spec}(\mathbb{S}) \xrightarrow{1} \mathbb{A}^1/\mathbb{G}_m \xleftarrow{0} B\mathbb{G}_m,$$

whose pushforwards on quasicoherent sheaves should correspond to the inclusion of the generic and special fibres respectively. In fact, this is the content of Theorem 2.2.10 in [MRT19], which states that there is a commutative diagram of symmetric monoidal left adjoints

$$\begin{tikzcd}
\mathrm{Sp} \arrow[d] & \mathrm{Sp}^{\mathrm{Fil}} \arrow[l, "\mathrm{Re}"'] \arrow[r, "\mathrm{gr}"] \arrow[d] & \mathrm{Sp}^{\mathrm{Gr}} \arrow[d] \\
\mathrm{QCoh}(\mathrm{Spec}(\mathbb{S})) & \mathrm{QCoh}(\mathbb{A}^1/\mathbb{G}_m) \arrow[l, "1^*"'] \arrow[r, "0^*"] & \mathrm{QCoh}(B\mathbb{G}_m),
\end{tikzcd}$$

where the vertical arrows are the previously established equivalences. In particular, this tells us that the realisation and associated graded functors we constructed earlier arise from this geometric picture as well. We will see later that these fibres of the stack $\mathbb{A}^1/\mathbb{G}_m$ and the left adjoints associated to their inclusions can be tensored up to more general deformations to determine their generic and special fibres as well. For now, let us recall the simple observation

$$\mathrm{Sp} \simeq \mathrm{Sp}^{\mathrm{Fil}}[\tau^{-1}] \subset \mathrm{Sp}^{\mathrm{Fil}}, \qquad \mathrm{Sp}^{\mathrm{Gr}} \simeq \mathrm{Mod}(\mathrm{Sp}^{\mathrm{Fil}}; C\tau)$$

from our discussion on filtered spectra to extend this commutative diagram to

$$\begin{tikzcd}
\mathrm{Sp}^{\mathrm{Fil}}[\tau^{-1}] \arrow[d] & \mathrm{Sp}^{\mathrm{Fil}} \arrow[l, "\tau^{-1}"'] \arrow[r, "C\tau\otimes-"] \arrow[d] & \mathrm{Mod}(\mathrm{Sp}^{\mathrm{Fil}}; C\tau) \arrow[d] \\
\mathrm{Sp} \arrow[d] & \mathrm{Sp}^{\mathrm{Fil}} \arrow[l, "\mathrm{Re}"'] \arrow[r, "\mathrm{gr}"] \arrow[d] & \mathrm{Sp}^{\mathrm{Gr}} \arrow[d] \\
\mathrm{QCoh}(\mathrm{Spec}(\mathbb{S})) & \mathrm{QCoh}(\mathbb{A}^1/\mathbb{G}_m) \arrow[l, "1^*"'] \arrow[r, "0^*"] & \mathrm{QCoh}(B\mathbb{G}_m),
\end{tikzcd}$$

where the vertical arrows are still equivalences. This translation of the geometric picture in terms of the thread operator τ will be helpful in describing more general deformations as will be done in the following section.

7.2 Geometric deformations

Inspired by the previous section, in which we saw that the geometric stack $\mathbb{A}_1/\mathbb{G}_m$ controls one-parameter deformations with a special fibre and a generic fibre, as well as the observation that $\mathrm{QCoh}(\mathbb{A}^1/\mathbb{G}_m) \simeq \mathrm{Sp}^{\mathrm{Fil}}$, we give a definition of what a deformation of homotopy theories should be based on geometric intuition.

Definition 7.1. A **geometric deformation** is a stable presentably symmetric monoidal ∞-category with the structure of an $\mathrm{Sp}^{\mathrm{Fil}}$-algebra in $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}$. Additionally, we will assume that deformations are $\mathbb{Z}$-plurigenic, i.e. admit a compact generator with respect to $\mathrm{Sp}^{\mathrm{Fil}}$-enriched mapping objects.

Remark 7.2. Just as in abelian groups, where an algebra object in the category of modules over a base ring is equivalent to an algebra object in abelian groups admitting a ring map from said base ring, we have a similar result in $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}$, namely [Lur17] Corollary 3.4.1.7. This tells us that

$$\mathrm{CAlg}(\mathrm{Mod}(\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}; \mathrm{Sp}^{\mathrm{Fil}})) \simeq \mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}})_{\mathrm{Sp}^{\mathrm{Fil}}/}.$$

In particular, we can think of deformations of stable homotopy theories simply as stable homotopy theories admitting a symmetric monoidal left adjoint from $\mathrm{Sp}^{\mathrm{Fil}}$

Remark 7.3. Note that the definition is a direct translation of the geometric picture. Indeed, if a one-parameter deformation of stacks is a family sitting over $\mathbb{A}^1/\mathbb{G}_m$, then applying the functor

$$\mathrm{QCoh}^* : \mathcal{S}\mathrm{hv}_{\mathrm{fpqc}}(\mathrm{CAlg}^{\mathrm{cn}}) \to \mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}$$

that sends a spectral stack to its ∞-category of quasicoherent sheaves and morphisms f to their pullbacks f^*–precisely the embedding of stacks into noncommutative stacks–should send it to a deformation. More precisely, some

$$f : \mathcal{X} \to \mathbb{A}^1/\mathbb{G}_m$$

gets sent to the deformation of homotopy theories

$$f^* : \mathrm{Sp}^{\mathrm{Fil}} \to \mathrm{QCoh}(\mathcal{X}).$$

Remark 7.4. Note that the assumption that $\mathcal{D}$ be $\mathbb{Z}$-plurigenic is not strictly necessary from the geometric picture, but it is very useful for identifying the generic and special fibres of a deformation. Concretely, if $\mathbb{1}_{\mathcal{D}}$ is the compact generator for $\mathcal{D}$ in an appropriately enriched sense, then

$$\rho : \mathcal{D} \to \mathrm{Sp}^{\mathrm{Fil}}$$

becomes conservative. Indeed, since λ is symmetric monoidal, it corresponds to (the $\mathrm{Sp}^{\mathrm{Fil}}$-linearisation) of the map picking out the object $\mathbb{1}_{\mathcal{D}}$ in $\mathcal{D}$, so that its right adjoint is given by

$$\rho = \mathrm{Map}^{\mathbb{Z}}(\mathbb{1}_{\mathcal{D}}, -).$$

By the filtered Schwede–Schipley theorem, this allows us to factor $\lambda \dashv \rho$ as an equivalence

$$\mathcal{D} \simeq \mathrm{Mod}(\mathrm{Sp}^{\mathrm{Fil}}; \mathrm{End}^{K}_{\mathcal{D}}(\mathbb{1}_{\mathcal{D}})).$$

Given this geometric intuition, it is hopefully clear why we have given such a precise construction and description of filtered spectra: they are the universal deformation of homotopy theories. Further, by the definition above, they control all other deformations. In fact, many notions intrinsic to deformations of homotopy theories are simply "tensored up" from filtered spectra as will be made rigorous later. The author's hope is that this geometric description–albeit not entirely rigorous due to the theory of spectral algebraic geometry being too hefty to summarise in this work–gives the reader an intuitive idea of why filtered objects are omnipresent in these *deformed* homotopy theories.

Let $\mathcal{D}$ now be a deformation of homotopy theories, and let

$$\lambda : \mathrm{Sp}^{\mathrm{Fil}} \rightleftarrows \mathcal{D} : \rho$$

be the symmetric monoidal left adjoint and its (lax monoidal) right adjoint witnessing this structure. We immediately note the following things.

- By monoidality, we see that $\lambda(\mathbb{1}_{\mathrm{Fil}}) \simeq \mathbb{1}_{\mathcal{D}}$. Now we extend this by defining

  $$\forall n \in \mathbb{Z} : \mathbb{1}_{\mathcal{D}}(n) := \lambda(\mathbb{1}_{\mathrm{Fil}}(n)).$$

  Since λ is symmetric monoidal, it is clear that we recover the relation

  $$\mathbb{1}_{\mathcal{D}}(n) \otimes \mathbb{1}_{\mathcal{D}}(m) \simeq \lambda(\mathbb{1}_{\mathrm{Fil}}(n) \otimes \mathbb{1}_{\mathrm{Fil}}(m)) \simeq \mathbb{1}_{\mathcal{D}}(n+m).$$

  In particular, all of these *twists* of the unit are dualisable objects.

- Recall that filtered spectra have a thread structure

  $$\tau : \mathbb{1}_{\mathrm{Fil}}(-1) \to \mathbb{1}_{\mathrm{Fil}}.$$

  We denote the image of this under λ by $\tau_{\mathcal{D}}$.

- Since λ commutes with colimits, we see that

  $$C\tau_{\mathcal{D}} \simeq \lambda(\mathrm{cof}(\mathbb{1}_{\mathrm{Fil}}(-1) \xrightarrow{\tau} \mathbb{1}_{\mathrm{Fil}}) \simeq \lambda(C\tau),$$

  so that $C\tau_{\mathcal{D}}$ has the structure of a commutative algebra object in $\mathcal{D}$ by monoidality of λ.

- This allows us to define subcategories of $\mathcal{D}$ just as in filtered spectra:

  $$\mathcal{D}[\tau_{\mathcal{D}}^{-1}], \qquad \mathrm{Mod}(\mathcal{D}, C\tau_{\mathcal{D}}), \qquad \mathcal{D}^{\wedge}_{\tau_{\mathcal{D}}}.$$

  Where the first and last full subcategories participate in a recollement

  $$\begin{tikzcd} \mathcal{D}[\tau_{\mathcal{D}}^{-1}] \arrow[r] & \mathcal{D} \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathcal{D}^{\wedge}_{\tau_{\mathcal{D}}} \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}.$$

Essential to the geometric picture is that the notion of the special and generic fibres of a deformation are given quite naturally.

Definition 7.2. Let $\mathcal{D}$ be a deformation of homotopy theories, as exhibited by the symmetric monoidal left adjoint

$$\lambda : \mathrm{Sp}^{\mathrm{Fil}} \to \mathcal{D}.$$

Now consider the right adjoints induced by the inclusion of the generic resp. special fibre

$$\mathrm{Sp} \xrightarrow{Cs} \mathrm{Sp}^{\mathrm{Fil}} \xleftarrow{P} \mathrm{Sp}^{\mathrm{Gr}}$$

We then define the **generic fibre** $\mathrm{GF}(\mathcal{D})$ and **special fibre** $\mathrm{SF}(\mathcal{D})$ of the deformation $\mathcal{D}$ to be the fibre products (taken in $\mathcal{P}r^{\mathrm{R}}_{\mathrm{St}}$, ergo equivalently in $\mathcal{C}at_\infty$ by [Lur09] Theorem 5.5.3.18)

$$\begin{tikzcd} \mathrm{GF}(\mathcal{D}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{D} \arrow[d, "\rho"] \\ \mathrm{Sp} \arrow[r, "\mathrm{Cs}"] & \mathrm{Sp}^{\mathrm{Fil}}, \end{tikzcd} \qquad \begin{tikzcd} \mathrm{SF}(\mathcal{D}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{D} \arrow[d, "\rho"] \\ \mathrm{Sp}^{\mathrm{Gr}} \arrow[r, "P"] & \mathrm{Sp}^{\mathrm{Fil}}. \end{tikzcd}$$

Geometrically, these are the fibres of the noncommutative stack $\mathcal{D}$ over $\mathrm{Spec}(\mathbb{S})$ and $B\mathbb{G}_m$ respectively, whence the terminology.

Remark 7.5. Since the functor Cs is a fully faithful functor arising as the inclusion of a full subcategory, and these are closed under pullbacks in $\mathcal{C}at_\infty$, we see that the generic fibre can be seen as a full subcategory of $\mathcal{D}$.

Proposition 7.3. *There is an equivalence*

$$\mathrm{GF}(\mathcal{D}) \simeq \mathcal{D}[\tau_{\mathcal{D}}^{-1}]$$

between the generic fibre of the deformation $\mathcal{D}$ and the τ-invertible objects.

Proof. This is now a rather immediate consequence of our setup. Indeed, ρ is a conservative functor, being constructed as the enriched mapping object out of the unit in a $\mathbb{Z}$-plurigenic homotopy theory.

Let X be some object of $\mathcal{D}$, with the map

$$X \otimes \tau_{\mathcal{D}} : X(-1) := X \otimes \mathbb{1}_{\mathcal{D}}(-1) \to X.$$

Since $\mathbb{1}_{\mathcal{D}}$ is an $\mathrm{Sp}^{\mathrm{Fil}}$-enriched generator, we see that $X \otimes \tau$ is an equivalence if and only if

$$\rho(X \otimes \tau) : \rho(X(-1)) = \mathrm{Map}^{\mathbb{Z}}(\mathbb{1}_{\mathcal{D}}, X(-1)) \to \mathrm{Map}^{\mathbb{Z}}(\mathbb{1}_{\mathcal{D}}, X)$$

is an equivalence. However, note that the $\mathrm{Sp}^{\mathrm{Fil}}$-enriched mapping objects are defined by

$$\mathrm{Map}^{\mathbb{Z}}(\mathbb{1}_{\mathcal{D}}, X)_\star \simeq \mathrm{Map}(\mathbb{1}_{\mathcal{D}}(\star^{\mathrm{op}}), X),$$

so that $\rho(X \otimes \tau)$ is simply the shift map on the filtered spectrum $\mathrm{Map}^{\mathbb{Z}}(\mathbb{1}_{\mathcal{D}}, X)$. This means that $X \otimes \tau$ is an equivalence if and only if the filtered spectrum

$$\mathrm{Map}^{\mathbb{Z}}(\mathbb{1}_{\mathcal{D}}, X) = \rho(X)$$

is τ-invertible, i.e. lies in the full subcategory $\mathrm{Sp} \hookrightarrow \mathrm{Sp}^{\mathrm{Fil}}$. This is precisely equivalent to the condition that X lies in the pullback defining the generic fibre, whence we conclude.

Having checked this explicit description of the pullback on the level of objects, we do not need to check it on mapping spaces, since we showed earlier that the generic fibre is a full subcategory of the deformation, just like the τ-invertible subcategory. ◻

Proposition 7.4. *There is an equivalence*

$$\mathrm{SF}(\mathcal{D}) \simeq \mathrm{Mod}(\mathcal{D}; C\tau)$$

between the special fibre of the deformation $\mathcal{D}$ and the $C\tau$-modules.

Proof. Once again, this follows from the definition of the special fibre as a pullback in $\mathcal{P}r^{\mathrm{R}}_{\mathrm{St}}$. Note that by duality, we can also compute this as a pushout in $\mathcal{P}r^{\mathrm{L}}_{\mathrm{St}}$. Once again, we need the assumption that $\mathcal{D}$ is $\mathbb{Z}$-plurigenic to apply the filtered Schwede–Shipley theorem and obtain

$$\mathcal{D} \simeq \mathrm{Mod}(\mathrm{Sp}^{\mathrm{Fil}}; \mathrm{End}^{\mathbb{Z}}_{\mathcal{D}}(\mathbb{1}_{\mathcal{D}})).$$

For brevity, we shall denote $\mathrm{End}^{\mathbb{Z}}_{\mathcal{D}}(\mathbb{1}_{\mathcal{D}})$ by R. Noting that

$$\mathrm{Sp}^{\mathrm{Gr}} \simeq \mathrm{Mod}(\mathrm{Sp}^{\mathrm{Fil}}; C\tau),$$

we see that the defining diagram for the special fibre can be rewritten as

$$\begin{tikzcd}
\mathrm{Sp}^{\mathrm{Fil}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathrm{Mod}(\mathrm{Sp}^{\mathrm{Fil}}; C\tau) \arrow[d] \\
\mathrm{Mod}(\mathrm{Sp}^{\mathrm{Fil}}; R) \arrow[r] & \mathrm{SF}(\mathcal{D}).
\end{tikzcd}$$

This pushout is now computed in $\mathcal{P}r^{\mathrm{L}}_{\mathrm{St}}$, or even in $\mathrm{Mod}(\mathcal{P}r^{\mathrm{L}}_{\mathrm{St}}; \mathrm{Sp}^{\mathrm{Fil}})$. At this point, we need some more advanced machinery. This comes in the form of [LT19] Theorem 1.10, or [Lur17] Proposition 7.1.2.6. This tells us that there is a fully faithful functor

$$\mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sp}^{\mathrm{Fil}}) \to \mathrm{Alg}_{\mathbb{E}_0}(\mathrm{Mod}(\mathcal{P}r^{\mathrm{L}}_{\mathrm{St}}; \mathrm{Sp}^{\mathrm{Fil}})),$$

with right adjoint (and partial inverse) sending an element of the target, which one can easily verify consists of an $\mathrm{Sp}^{\mathrm{Fil}}$-module $\mathcal{M}$ with a distinguished object m, to the $\mathbb{E}_1$-algebra $\mathrm{End}^{\mathbb{Z}}_{\mathcal{M}}(m)$. Finally, note that there is a forgetful functor

$$\mathrm{Alg}_{\mathbb{E}_0}(\mathrm{Mod}(\mathcal{P}r^{\mathrm{L}}_{\mathrm{St}}; \mathrm{Sp}^{\mathrm{Fil}})) \to \mathrm{Mod}(\mathcal{P}r^{\mathrm{L}}_{\mathrm{St}}; \mathrm{Sp}^{\mathrm{Fil}}))$$

that commutes with pushouts. More precisely, cf. [Lur17] Corollary 4.8.5.13, it commutes with colimits indexed by weakly contractible simplicial sets. We conclude that the pushout in the latter can be lifted to a pushout in the source by equipping each of the terms in the puhsout diagram with the structure of an $\mathbb{E}_0$-algebra simply by marking it at the unit $\mathbb{1}_{\mathcal{D}}$, resp. $\mathbb{1}_{\mathrm{Gr}}$. Since this lies in the image of the embedding of $\mathbb{E}_1$-algebras in filtered spectra into the source of the forgetful map, we can once again lift these and identify them with the algebras $R, C\tau$ in $\mathrm{Sp}^{\mathrm{Fil}}$ respectively. Now one can simply take the pushout in this ∞-category, and run our (pushout-preserving) functors the right way to obtain a description of the pushout of $\mathrm{Sp}^{\mathrm{Fil}}$-modules. Identifying the pushout of the $\mathbb{E}_\infty$-algebra $C\tau$ and the $\mathbb{E}_1$-algebra R over $\mathbb{1}_{\mathrm{Fil}}$ can be tricky a priori, but the commutativity of $C\tau$ drastically simplifies the situation. Indeed, it allows us to apply Theorem 3.6 from [HL21] in the case $k = 1$. This theorem essentially tells us that the pushout

$$C\tau \sqcup_{\mathbb{1}_{\mathrm{Fil}}} R$$

in $\mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sp}^{\mathrm{Fil}})$ can be identified (as an $\mathbb{E}_1$-algebra) with the tensor product

$$C\tau \otimes_{\mathbb{1}_{\mathrm{Fil}}} R.$$

Concretely, we obtain

$$\mathrm{SF}(\mathcal{D}) \simeq \mathrm{Mod}(\mathrm{Sp}^{\mathrm{Fil}}; R \otimes_{\mathbb{1}_{\mathrm{Fil}}} C\tau).$$

This ∞-category of $(R, C\tau)$-bimodules can finally be identified with

$$\mathrm{SF}(\mathcal{D}) \simeq \mathrm{Mod}(\mathrm{Mod}(\mathrm{Sp}^{\mathrm{Fil}}; R); C\tau)) \simeq \mathrm{Mod}(\mathcal{D}; C\tau)$$

to obtain the desired result. □

7.3 BHS deformations

Having given a geometric motivation for what deformations of homotopy theories (a.k.a. noncommutative stacks) should be, we proceed to give a more practical description of deformations in terms of a realisation functor and a choice of graded objects. This approach is described by Burklund–Hahn–Senger in [BHS20], and we will only slightly adapt their approach. We end this section with an equivalence between Burklund–Hahn–Senger's deformations and our geometric deformations. This allows us to use both frameworks interchangeably.

Definition 7.3 ([BHS20], C.13)**.** A BHS deformation is a diagram

$$\mathcal{C} \xrightarrow{c} \mathcal{D} \xrightarrow{\mathrm{Re}} \mathcal{C}$$

in $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}})$ such that the composite $\mathrm{Re} \circ c$ is the identity. Additionally, we require the datum for every $n \in \mathbb{Z}$ of an invertible element $\mathbb{1}_{\mathcal{D}}(n)$ in $\mathcal{D}$ that gets sent to the unit $\mathbb{1}_{\mathcal{C}}$. These should assemble to a group homomorphism

$$\mathbb{Z} \to \ker \mathrm{Pic}_0 \mathrm{Re},$$

in the sense that for every $n, m \in \mathbb{Z}$ we should have an equivalence

$$\mathbb{1}_{\mathcal{D}}(n) \otimes \mathbb{1}_{\mathcal{D}}(m) \simeq \mathbb{1}_{\mathcal{D}}(n+m).$$

and in particular $\mathbb{1}_{\mathcal{D}}(0) \simeq \mathbb{1}_{\mathcal{D}}$. Finally, we require that Re induces an equivalence on mapping spaces

$$\mathrm{map}_{\mathcal{D}}(\mathbb{1}_{\mathcal{D}}(n), \mathbb{1}_{\mathcal{D}}(m)) \xrightarrow{\sim} \mathrm{map}_{\mathcal{C}}(\mathbb{1}_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}})$$

for every $n \leq m \in \mathbb{Z}$

Remark 7.6. Note that the last condition can be applied for $n = 0 \leq 0 = m$ to obtain a series of equivalences

$$\mathrm{map}_{\mathcal{D}}(\mathbb{1}_{\mathcal{D}}(n), \mathbb{1}_{\mathcal{D}}(m)) \simeq \mathrm{map}_{\mathcal{C}}(\mathbb{1}_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}}) \simeq \mathrm{map}_{\mathcal{D}}(\mathbb{1}_{\mathcal{D}}, \mathbb{1}_{\mathcal{D}}).$$

In fact, this allows us to rewrite the data of the invertible elements in a more succinct way, i.e. as an increasingly filtered object

$$\mathbb{1}(\star^{\mathrm{op}}) : \mathbb{Z} \to \mathcal{D}$$

such that the composite $\mathrm{Re}\mathbb{1}(\star^{\mathrm{op}})$ is the constant filtered object $\mathrm{Cs}\mathbb{1}_{\mathcal{C}}$.

Remark 7.7. In most cases of interest, the base category $\mathcal{C}$ to be deformed is precisely the ∞-category of spectra Sp. In that case, since Sp is the initial object of $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}$ and the composite $\mathrm{Re} \circ c$ of left adjoints would be a left adjoint from Sp to itself, we conclude that it is automatically equivalent to the identity.

This definition is expressed entirely in terms of the datum of two symmetric monoidal left adjoint functors and a compatible datum of invertible objects, so that it is useful in identifying deformations of homotopy theories in the wild. However, as also elaborated in [BHS20], we see that the datum of a deformation as above actually recovers the more formal notion of a deformation in terms of filtered spectra. Indeed, one can upgrade $\mathcal{D}$ to an $\mathrm{Sp}^{\mathrm{Fil}}$-algebra in $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}$ with generic fibre $\mathcal{C}$ and the local filtration given precisely by the shifts of the unit $\mathbb{1}_{\mathcal{D}}(n)$.

Theorem 7.1. *The datum of a deformation as in Definition 7.3 gives rise to the structure of a deformation on $\mathcal{D}$ i.e. $\mathcal{D}$ obtains the structure of a $\mathrm{Sp}^{\mathrm{Fil}}$-algebra in $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}$.*

Proof. First, let us recall that due to the equivalence $\mathcal{C}^{\mathrm{Fil}} \simeq \mathrm{Sp}^{\mathrm{Fil}} \otimes \mathcal{C}$, we can reduce to showing that $\mathcal{D}$ is an $\mathrm{Sp}^{\mathrm{Fil}}$-algebra, and then use the given left adjoint $c : \mathcal{C} \to \mathcal{D}$ to tensor this up to a $\mathrm{Sp}^{\mathrm{Fil}} \otimes \mathcal{C} \simeq \mathcal{C}^{\mathrm{Fil}}$-algebra.

Now recall that the datum of an $\mathrm{Sp}^{\mathrm{Fil}}$-algebra structure on $\mathcal{D}$ is equivalent to the datum of a symmetric monoidal left adjoint $\mathrm{Sp}^{\mathrm{Fil}} \to \mathcal{D}$. Further, let us note that $\mathrm{Sp}^{\mathrm{Fil}} = \mathcal{F}\mathrm{un}(\mathbb{Z}^{\mathrm{op}}, \mathrm{Sp})$ is obtained as the stabilisation of the presheaf category $\mathcal{F}\mathrm{un}(\mathbb{Z}^{\mathrm{op}}, \mathcal{S})$, which itself is obtained as the colimit completion of $\mathbb{Z}$. Additionally, recall that the symmetric monoidal structures on stabilisations or colimit completions of (small) ∞-categories were obtained precisely by the universal property that the corresponding left adjoints to inclusions be symmetric monoidal. Therefore, one can reduce the datum of a symmetric monoidal left adjoint of presentable stable ∞-categories of the form

$$\mathrm{Sp}^{\mathrm{Fil}} \simeq \mathrm{Sp}(\mathcal{F}\mathrm{un}(\mathbb{Z}^{\mathrm{op}}, \mathcal{S})) \simeq \mathrm{Sp}(\mathcal{P}(\mathbb{Z})) \to \mathcal{D}$$

to the datum of a symmetric monoidal functor

$$\mathbb{Z} \to \mathcal{D}.$$

This will be provided precisely by the data of the shifts $\mathbb{1}_{\mathcal{D}}(n)$. Indeed, the final condition in Definition 7.3 and the remark below gives us an equivalence

$$\mathrm{map}_{\mathcal{D}}(\mathbb{1}_{\mathcal{D}}(n), \mathbb{1}_{\mathcal{D}}(m)) \simeq \mathrm{map}_{\mathcal{D}}(\mathbb{1}_{\mathcal{D}}, \mathbb{1}_{\mathcal{D}}).$$

Therefore, for any morphism $n \leq m$ in $\mathbb{Z}$, we let the morphism from $\mathbb{1}_{\mathcal{D}}(n)$ to $\mathbb{1}_{\mathcal{D}}(m)$ be the one corresponding to the identity of $\mathbb{1}_{\mathcal{D}}$ in the equivalence above. At this point, we need to be careful with the ∞-category theory going on, since we are actually considering functors from the ∞-category $N\mathbb{Z}$ into $\mathcal{D}$. This means that it does not suffice to simply define morphism corresponding to every $n \leq m$, but that one also ought to define composite morphisms and compatible homotopies between them expressing them as composites. However, we are able to get away with the simple construction above, since the obvious map of simplicial sets

$$L := \bigcup_{q \in \mathbb{Z}} \Delta^{\{q-1,q\}} \to N\mathbb{Z}$$

classifying the morphisms between adjacent integers is inner anodyne (cf. [Ari21] Prop 3.3), so that the morphism

$$L \to \mathcal{D}$$

defined by $(n \leq n+1) \mapsto (\mathbb{1}_{\mathcal{D}}(n) \to \mathbb{1}_{\mathcal{D}}(n+1))$ extends uniquely to a morphism

$$N\mathbb{Z} \to \mathcal{D}$$

as desired. This is the content of Lemma 7.1. Having established this functor, it is clear that it is symmetric monoidal per construction since we enforced $\mathbb{1}_{\mathcal{D}}(n) \otimes \mathbb{1}_{\mathcal{D}}(m) \simeq \mathbb{1}_{\mathcal{D}}(n+m)$, and all morphisms between them correspond to the identity on $\mathbb{1}_{\mathcal{D}}$. We conclude that it induces a symmetric monoidal left adjoint $\mathrm{Sp}^{\mathrm{Fil}} \to \mathcal{D}$ as desired. □

Lemma 7.1. *The aforementioned inclusion $L \to N\mathbb{Z}$ is an inner anodyne map of simplicial sets.*

Proof. The fact that this inclusion is inner anodyne can be seen by observing that $N\mathbb{Z}$ is obtained from L by adjoining coherent composites. Concretely, if one lets S_p denote the subsimplicial set of L defined by

$$S_p := \bigcup_{-p \leq q-1 \leq q \leq p} \Delta^{\{q-1,q\}},$$

viewed as the spine $S_p = \mathrm{spine}(\Delta^{[-p,p]})$, then one can consider the gluing

$$\begin{tikzcd} S_p \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \Delta^{\{-p,p\}} \arrow[d] \\ L \arrow[r] & G_p, \end{tikzcd}$$

where the interval S_p has been replaced by the simplex $\Delta^{[-p,p]}$, i.e. we have adjoined all coherent composites. Then we obtain $N\mathbb{Z}$ as the directed colimit

$$N\mathbb{Z} \simeq \operatorname*{colim}_{p \geq 0} G_p.$$

It then suffices to note that in the pushout

$$\begin{tikzcd}
\Delta^{\{-p-1,-p\}} \cup \Delta^{[-p,p]} \cup \Delta^{\{p,p+1\}} \arrow[r] \arrow[d] & G_p \arrow[d] \\
\Delta^{[-p-1,p+1]} \arrow[r] \arrow[ur, phantom, "\ulcorner"] & G_{p+1},
\end{tikzcd}$$

the left vertical arrow is inner anodyne as a generalised spine inclusion, so that by cosaturation the inclusion map $G_p \to G_{p+1}$ is inner anodyne as well. We conclude our argument by noting that $G_0 \simeq L$ so that the inclusion $L \to N\mathbb{Z}$ is a transfinite composition of inner anodyne morphisms, hence inner anodyne itself. □

In fact, this lemma is very often used implicitly when constructing filtered objects.

Note that the theorem above only gives $\mathcal{D}$ the structure of a deformation, but does not necessarily determine the generic and special fibres of $\mathcal{D}$ in terms of the given information. The definition clearly hints at $\mathcal{C}$ being the generic fibre of $\mathcal{D}$, but for this to be true, we need to impose a slightly stricter condition on the deformation in the form of a conservativity (or monadicity) condition.

Proposition 7.5 ([BHS20], C.19)**.** *Let $\mathcal{C} \to \mathcal{D} \to \mathcal{C}$ be a BHS deformation as in Definition 7.3. If we further assume that $\mathcal{C}$ is generated by a family of compact dualisable objects G_α, such that the objects $c(G_\alpha) \otimes \mathbb{1}_{\mathcal{D}}(n)$ form a family of compact dualisable generators for $\mathcal{D}$, then the previously constructed adjunction*

$$\mathcal{C}^{\mathrm{Fil}} \rightleftarrows \mathcal{D}$$

is monadic.

Remark 7.8. In fact, this assumption (which is imposed on all deformations on op. cit.) corresponds precisely to a $\mathbb{Z}$-plurigenicity assumption on $\mathcal{D}$–albeit in in $\mathcal{C}$-linear form. Indeed, let us assume that $\mathcal{C} = \mathrm{Sp}$ for the moment, since all other cases can be tensored up from it. Then c is the essentially unique symmetric monoidal left adjoint picking out the unit of $\mathcal{D}$, so this amounts to asking that the twists $\mathbb{1}_{\mathcal{D}}(n)$ are a family of compact generators for $\mathcal{D}$ (they are clearly dualisable). Since $\mathrm{Sp}^{\mathrm{Fil}}$-enriched mapping objects in $\mathrm{Sp}^{\mathrm{Fil}}$ are obtained precisely by these twists, such that the right adjoint structure map

$$\rho = \mathrm{Map}^{\mathbb{Z}}(\mathbb{1}_{\mathcal{D}}, -) = \mathrm{Map}(\mathbb{1}_{\mathcal{D}}(\star^{\mathrm{op}}), -) : \mathcal{D} \to \mathrm{Sp}^{\mathrm{Fil}}$$

is given by mapping out of twists of the unit, we see that the requirement above is precisely that $\mathcal{D}$ be $\mathbb{Z}$-plurigenic. We can then apply our theory of $\mathbb{Z}$-plurigenic geometric deformations of homotopy theories to see that the filtered Schwede–Shipley theorem applies, and $\mathcal{D}$ can be realised as

$$\mathcal{D} \simeq \mathrm{Mod}(\mathrm{Sp}^{\mathrm{Fil}}; \mathrm{End}^{\mathbb{Z}}_{\mathcal{D}}(\mathbb{1}_{\mathcal{D}})) \simeq \mathrm{Mod}(\mathrm{Sp}^{\mathrm{Fil}}; \mathrm{Map}_{\mathcal{D}}(\mathbb{1}_{\mathcal{D}}(\star^{\mathrm{op}}), \mathbb{1}_{\mathcal{D}})).$$

Combining the main theorem above with the monadicity result below, we see that these two notions of deformations really are equivalent.

Corollary 7.1. *The structure of a geometric deformation on some stable presentably symmetric monoidal ∞-category $\mathcal{D}$ is equivalent to the datum of a BHS deformation.*

Proof. We saw in Theorem 7.1 that any BHS deformation is a geometric deformation in an essentially unique way. Further, we saw that the usual monadicity assumption on BHS deformations is none other than a $\mathbb{Z}$-plurigenicity condition on its associated geometric deformation.

Conversely, given the structure of a geometric deformation on $\mathcal{D}$, we can construct a BHS deformation by considering the diagram

$$\mathrm{GF}(\mathcal{D}) \xrightarrow{c} \mathcal{D} \xrightarrow{\mathrm{Re}} \mathrm{GF}(\mathcal{D}).$$

Indeed, we know that $\mathcal{D}$ is an $\mathrm{GF}(\mathcal{D})$-linear ∞-category by the construction of the latter, so that we simply consider the unit c of $\mathcal{D}$ viewed as an $\mathrm{GF}(\mathcal{D})$-algebra, along with the τ-inversion map Re left adjoint to its inclusion. The composite is equivalent to the identity because it a symmetric monoidal left adjoint functor of SF($\mathcal{D}$)-algebras, hence essentially unique. Now we choose the dualisable objects in $\mathcal{D}$ to be precisely the twists $\mathbb{1}_{\mathcal{D}}(n)$ induced by the structure of a geometric deformation. These are dualisable per construction, and satisfy the monoidality relation in the index n. Further, per definition of the generic fibre there is a commutative square of symmetric monoidal left adjoints

$$\begin{tikzcd} \mathcal{D} \arrow[r] & \mathrm{GF}(\mathcal{D}) \\ \mathrm{Sp}^{\mathrm{Fil}} \arrow[u] \arrow[r] & \mathrm{Sp}. \arrow[u] \end{tikzcd}$$

Starting with the twists $\mathbb{1}_{\mathrm{Fil}}(n)$ in the bottom left corner, we see that they are sent to the τ-inversion of $\mathbb{1}_{\mathcal{D}}(n)$ by one path, and send to the unit $\mathbb{1}_{\mathrm{GF}(\mathcal{D})}$ by the other. Indeed, we verified that these twists of the unit in filtered spectra realise to the sphere spectrum, and by monoidality this must be sent to the unit in the generic fibre. We conclude that these define elements in $\ker \mathrm{Pic}_0 \mathrm{Re}$.

In fact, since the generic fibre is a full subcategory of $\mathcal{D}$, and one easily verifies that $\mathrm{map}(\mathbb{1}_{\mathrm{Fil}}(n), \mathbb{1}_{\mathrm{Fil}}(m))$ is equivalent to $\mathrm{map}(\mathbb{S}, \mathbb{S})$ as induced by the realisation functor; we conclude that the condition on mapping spaces holds as well.

Finally, it is clear that these two operations are mutually inverse, whence one obtains an equivalence between geometric deformation structures on $\mathcal{D}$ and BHS deformation structures on $\mathcal{D}$. □

Remark 7.9. Note that most of the proof above just consisted of showing that $\mathrm{Sp}^{\mathrm{Fil}}$ is a deformation in the sense of [BHS20], and then tensoring up these results to an arbitrary geometric deformation.

Remark 7.10. Note that this picture of deformations does not give us an explicit characterisation of the special fibre. Instead, this must be recovered by viewing a BHS deformation as a geometric deformation, and recovering the special fibre as

$$\mathrm{SF}(\mathcal{D}) = \mathrm{Mod}(\mathcal{D}; C\tau).$$

Example 7.1. Let us illustrate the construction above with an example of a deformation that is not hard to construct. We claim that the ∞-category of cochain complexes of spectra $\mathcal{K}(\mathrm{Sp})$ introduced in Definition 5.4 admits the structure of a deformation of Sp. In fact, it can be described rather explicitly.

- Given an integer n, consider the invertible object of $\mathcal{K}(\mathrm{Sp})$ given by the cochain complex $\mathbb{1}_{\mathcal{K}}(n)$ that takes the value $\mathbb{S}^n$ in degree n and 0 elsewhere. It is clear that this defines a cochain complex.
- Now given integers m, n, note that the tensor product $\mathbb{1}_{\mathcal{K}}(n) \otimes \mathbb{1}_{\mathcal{K}}(m)$ of cochain complexes is such that

  $$(\mathbb{1}_{\mathcal{K}}(n) \otimes \mathbb{1}_{\mathcal{K}}(m))_r = \bigoplus_{r=p+q} \mathbb{1}_{\mathcal{K}}(n)_p \wedge \mathbb{1}_{\mathcal{K}}(m)_q = \begin{cases} \mathbb{S}^m \wedge \mathbb{S}^n, & r = m+n, \\ 0, & \text{else}, \end{cases}$$

  i.e. it is clearly equivalent to $\mathbb{1}_{\mathcal{K}}(m+n)$.
- If $m \leq n$, we see that the mapping space

  $$\mathrm{map}(\mathbb{1}_{\mathcal{K}}(m), \mathbb{1}_{\mathcal{K}}(n))$$

  can be described explicitly. When $m = n$, it is clear that one only needs to define a map on the nonzero degrees, so that

  $$\mathrm{map}(\mathbb{1}_{\mathcal{K}}(m), \mathbb{1}_{\mathcal{K}}(m)) \cong \mathrm{map}(\mathbb{S}^m, \mathbb{S}^m) \cong \mathrm{map}(\mathbb{S}, \mathbb{S}).$$

  If $n = m+1$, we see that the space of such maps is precisely the space of vertical maps making the diagram

  $$\begin{tikzcd} S^m \arrow[r, "\partial^{m+1}"] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r, "\partial^{m+1}"] & S^{m+1} \end{tikzcd}$$

commute, the rest being zero maps between zero objects. Now by the universal property of the pullback it suffices to give just one map from $\mathbb{S}^m$ to the pullback of this diagram, and the latter is precisely the loop $\Sigma^{-1}\mathbb{S}^{m+1} \simeq \mathbb{S}^m$, so that we conclude similarly as before. This procedure can then be iterated for any $m < n$, taking successive pullbacks until we need only define a map $\mathbb{S}^m \to \Sigma^{m-n}\mathbb{S}^n$, and we conclude that all mapping spaces of the form above are equivalent to the endomorphism space of the sphere spectrum.

Just the data above defines a functor

$$\mathbb{1}_{\mathcal{K}}(\star^{\mathrm{op}}) : \mathbb{Z} \to \mathcal{K}(\mathrm{Sp}),$$

and it is then a consequence of the reasoning above that one can integrate it to an adjunction

$$\lambda : \mathrm{Sp}^{\mathrm{Fil}} \rightleftarrows \mathcal{K}(\mathrm{Sp}) : \rho.$$

Note that the right adjoint is given by

$$\rho X_\star = \mathrm{Map}(\mathbb{1}_{\mathcal{K}}(\star^{\mathrm{op}}, X)),$$

while the left adjoint per construction sends the shifted filtered spectrum $\mathbb{1}_{\mathrm{Fil}}(n)$ to the cochain complex $\mathbb{1}_{\mathcal{K}}(n)$ defined above.

We will now give a more explicit description of the left adjoint. First, let us consider another functor

$$\Sigma^* \mathrm{gr}_* : \mathrm{Sp}^{\mathrm{Fil}} \to \mathrm{Sp}^{\mathrm{Gr}}.$$

In fact, this only defines a graded spectrum, but we see that there are connecting maps induced by the filtered structure. Indeed, consider the defining cofibre sequence associated to a filtered spectrum X

$$X_{n+1} \to X_n \to \mathrm{gr}_n X,$$

or equivalently after applying the left adjoint Σ^n:

$$\Sigma^n X_{n+1} \to \Sigma^n X_n \to \Sigma^n \mathrm{gr}_n X.$$

By the triangulation on a stable ∞-category, one can shift this to a cofibre sequence

$$\Sigma^n X \to \Sigma^n \mathrm{gr}_n X \to \Sigma^{n+1} X_{n+1}.$$

Now the final term in this sequence admits a map to $\Sigma^{n+1}\mathrm{gr}_{n+1}X$ as the image under Σ^{n+1} of the quotient map. Therefore, we see that there are connecting maps

$$\Sigma^n \mathrm{gr}_n X \to \Sigma^{n+1}\mathrm{gr}_{n+1} X.$$

If we now consider a composite of the form

$$\Sigma^n \mathrm{gr}_n X \to \Sigma^{n+1}\mathrm{gr}_{n+1} X \to \Sigma^{n+2}\mathrm{gr}_{n+2} X,$$

we can write this out more fully using the definition of the connecting maps as

$$\Sigma^n \mathrm{gr}_n X \to \Sigma^{n+1} X_{n+1} \to \Sigma^{n+1}\mathrm{gr}_{n+1} X \to \Sigma^{n+2} X_{n+2} \to \Sigma^{n+2}\mathrm{gr}_{n+2} X.$$

If we now focus on the middle three terms, forming the sequence

$$\Sigma^{n+1} X_{n+1} \to \Sigma^{n+1}\mathrm{gr}_{n+1} X \to \Sigma^{n+2} X_{n+1},$$

we see that this is the triangulated shift of the cofibre sequence

$$\Sigma^{n+1} X_{n+2} \to \Sigma^{n+1} X_{n+1} \to \Sigma^{n+1}\mathrm{gr}_{n+1} X.$$

Now this composite is obviously null since it is a cofibre sequence. We conclude that the original composite of connecting maps was zero as well. In conclusion, the functor

$$\Sigma^* \mathrm{gr}_* : \mathrm{Sp}^{\mathrm{Fil}} \longrightarrow \mathcal{K}(\mathrm{Sp})$$

lifts to a functor with values in cochain complexes of spectra. Its values on some simple filtered spectra are easy to understand. Indeed, consider the shifted unit $\mathbb{1}_{\mathrm{Fil}}(n)$ in filtered spectra. Since this filtration is composed almost entirely of identities save for one map $0 \to \mathbb{S}$ in filtration degree n, we see that

$$\Sigma^* \mathrm{gr}_*(\mathbb{1}(n)) = \begin{cases} \Sigma^n \mathrm{cof}(0 \to \mathbb{S}) \simeq \mathbb{S}^n, & * = n, \\ 0, & \text{else.} \end{cases}$$

and the result is the chain complex $\mathbb{1}_{\mathcal{K}}(n)$. If $m < n$ and we are given the canonical morphism

$$\tau^{n-m} : \mathbb{1}_{\mathrm{Fil}}(m) \longrightarrow \mathbb{1}_{\mathrm{Fil}}(n)$$

induced by inserting identities on the sphere spectra in filtration range between m and n, then the induced map on associated gradeds arises from the diagrams of the form below (illustrating the case $n = m + 2$)

$$\begin{tikzcd}
\cdots & \mathbb{S} \arrow[d] & \mathbb{S} \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
& \mathbb{S} \arrow[ul, "="] \arrow[d, dashed] & \mathbb{S} \arrow[ul, "="] \arrow[d, dashed] & \mathbb{S} \arrow[ul, "="] \arrow[d, dashed] & 0 \arrow[ul, "="] \arrow[d, dashed] & 0 \arrow[ul, "="] \arrow[d, dashed] & \cdots \arrow[ul, "="] \\
\cdots \arrow[r, dotted] & 0 \arrow[r, dotted] \arrow[d] & 0 \arrow[r, dotted] \arrow[d] & \mathbb{S}^m \arrow[r, dotted] \arrow[d] & 0 \arrow[r, dotted] \arrow[d] & 0 \arrow[r, dotted] \arrow[d] & \cdots \\
\cdots \arrow[r, dotted] & 0 \arrow[r, dotted] & 0 \arrow[r, dotted] & 0 \arrow[r, dotted] & 0 \arrow[r, dotted] & \mathbb{S}^{m+2} \arrow[r, dotted] & \cdots \\
& \mathbb{S} \arrow[u, dashed] \arrow[dl, "="] & \mathbb{S} \arrow[u, dashed] \arrow[dl, "="] & \mathbb{S} \arrow[u, dashed] \arrow[dl, "="] & \mathbb{S} \arrow[u, dashed] \arrow[dl, "="] & \mathbb{S} \arrow[u, dashed] \arrow[dl, "="] & \cdots \arrow[dl, "="] \\
\cdots & \mathbb{S} \arrow[u] & \mathbb{S} \arrow[u] & \mathbb{S} \arrow[u] & \mathbb{S} \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

in which the top and bottom rows represent the filtered spectra $\mathbb{1}(m)$ and $\mathbb{1}(m+2)$ respectively, running from right to left. They are depicted such that the long vertical dashed arrows represent the maps onto the cofibres (albeit shifted up in every degree). Then the dotted horizontal arrows are the structure maps of the corresponding cochain complexes, while the short vertical arrows constitute the map of cochain complexes obtained by functoriality. The observation to make here is that the induced map of cochain complexes

$$\mathbb{1}_{\mathcal{K}}(m) \longrightarrow \mathbb{1}_{\mathcal{K}}(n)$$

is none other than the map which is induced by the identity of $\mathbb{S}^m$, seen as a map from $\mathbb{S}^m$ to the $(n-m)$-fold iterated pullback of $\mathbb{S}^n$ along the trivial spectrum, i.e. it is none other than the original structure map between shifted units that was part of the deformation structure.

The conclusion of this discussion is that the functors

$$\lambda, \Sigma^* \mathrm{gr}_* : \mathrm{Sp}^{\mathrm{Fil}} \longrightarrow \mathcal{K}(\mathrm{Sp})$$

agree on the full subcategory of generators $\mathbb{1}_{\mathcal{K}}(n)$. Since they are both left adjoints, either by construction or since Σ and gr are left adjoints, we conclude that they agree on all filtered spectra, whence we have obtained an explicit description of λ.

7.4 Deformations and recollements

Now that we have two pictures of what a deformation should be, and have shown that they are equivalent, we would like to obtain a method of describing these deformations in terms of their special and generic fibres. This reconstruction will come in the form of a recollement associated to any deformation

In fact, since we have a geometric picture of deformations arising from our consideration in terms of noncommutative stacks over $\mathbb{A}^1/\mathbb{G}_m$, we can describe the recollement associated to a deformation in a geometric way. For this, let us recall that a recollement is classically induced by the inclusion of a closed subscheme and its open complement. We can consider the inclusion

$$B\mathbb{G}_m \hookrightarrow \mathbb{A}^1/\mathbb{G}_m$$

of the closed substack $B\mathbb{G}_m$ as a stratification along $[0 \leq 1]$, to obtain a corresponding stratification on $\mathrm{QCoh}(\mathbb{A}^1/\mathbb{G}_m) \simeq \mathrm{Sp}^{\mathrm{Fil}}$.

First, let us note that any deformation has a thread operator τ shifting the grading in its natural filtration. In the geometric deformation picture, i.e. for $\mathrm{Sp}^{\mathrm{Fil}}$-algebras, this τ was obtained from the thread operator on $\mathrm{Sp}^{\mathrm{Fil}}$, and similarly for BHS deformations. In particular, this means that we can mimic the deformation that arose in our discussion of filtered spectra, namely

$$\begin{tikzcd} \mathrm{Sp} \arrow[r] & \mathrm{Sp}^{\mathrm{Fil}} \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathrm{Sp}^{\mathrm{Fil}\wedge}_{\tau} \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}.$$

Note that this recollement arises purely from the datum of the thread operator τ, and corresponds to the decomposition in the τ-invertible and τ-complete subcategories.

Geometrically, recall that we identified

$$\mathrm{QCoh}(\mathrm{Spec}(\mathbb{S})) \simeq \mathrm{Sp}, \qquad \mathrm{QCoh}(B\mathbb{G}_m) \simeq \mathrm{Sp}^{\mathrm{Gr}}.$$

As explained in the beginning of this section, we are interested in the recollement that arises from the open-closed decomposition

$$B\mathbb{G}_m \hookrightarrow \mathbb{A}_1/\mathbb{G}_m \hookleftarrow \mathrm{Spec}(\mathbb{S}).$$

Now the open part is easy to identify. Indeed, we saw in the discussion of the geometry of filtrations that the restriction to this open substack corresponded precisely to the realisation or τ-inversion of a filtration. We conclude that the resulting recollement has open part given by the (co)reflective subcategory

$$\mathrm{Sp} \simeq \mathrm{Sp}^{\mathrm{Fil}}[\tau^{-1}] \subset \mathrm{Sp}^{\mathrm{Fil}}.$$

Now the closed part of the recollement is a little harder to identify, since it is not given by quasicoherent sheaves on the closed subscheme, but by a formal completion of quasicoherent sheaves on the total scheme along the closed subscheme. Since we do not want to describe the theory of completions along closed substacks in spectral algebraic geometry, we decide to circumvent this. Indeed, by Proposition 6.3, only one closed part can fit in this recollement, and it is the orthogonal complement. In the discussion on filtered spectra, we computed that the orthogonal complement of the τ-invertible filtered spectra were the complete filtrations. Therefore, we see that the recollement arising from the geometric picture is precisely the same one we constructed earlier.

Proposition 7.6. *Let $\mathrm{Sp}^{\mathrm{Fil}}$ be the stable presentably symmetric monoidal ∞-category of filtered spectra. Viewing this as a category of filtrations, the universal noncommutative stack with a dualisable commutative algebra object $C\tau$, and as quasicoherent sheaves on $\mathbb{A}^1/\mathbb{G}_m$, we obtain three recollements*:

- *A recollement where the open part consists of constant filtrations on their colimit, and the closed part consists of complete filtrations.*
- *A recollement where the open part if the τ-invertible subcategory, and the closed part is the τ-complete subcategory.*

- *A recollement where the open part is the subcategory of quasicoherent sheaves on Spec*($\mathbb{S}$)*, and the closed part is the completion of* $\mathrm{Sp}^{\mathrm{Fil}}$ *along quasicoherent sheaves on* $B\mathbb{G}_m$*.*

These recollements all agree.

Proof. This is now an immediate consequence of the discussion above. □

The use of this observation is that it makes it easier to construct recollements on deformations. Indeed, as noncommutative stacks over $\mathbb{A}^1/\mathbb{G}_m$, they can be decomposed according to the generic fibre and their formal completion along the special fibre, but does not admit any sort of intuitive description. We therefore opt to use the second recollement as the canonical one, and decompose deformations along their τ-invertible and τ-complete parts.

Definition 7.4. Given a deformation $\mathcal{D}$ with thread operator $\tau_{\mathcal{D}}$, let us define its associated recollement to be the recollement associated to the dualisable commutative algebra object $C\tau_{\mathcal{D}}$, i.e.

$$\begin{tikzcd} \mathcal{D}[\tau_{\mathcal{D}}^{-1}] \simeq \mathrm{GF}(\mathcal{D}) \arrow[r] & \mathcal{D} \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathcal{D}^{\wedge}_{\tau_{\mathcal{D}}} \simeq \mathcal{D}^{\wedge}_{\mathrm{SF}(\mathcal{D})} \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}.$$

In the notation above, we have used the fundamental result that the generic resp. special fibres of a deformation can be recovered as the τ-invertible objects resp. the $C\tau$-modules.

The value of this decomposition is that it allows us to solidify the idea of a deformation being the datum of a generic fibre and a special fibre, along with some gluing information that encodes how they are stuck together. In particular, we can apply our reconstruction theorem from the theory of recollements to obtain a reconstruction

8 Synthetic spectra

In this section, we come to the most important example of a deformation of homotopy theories that motivated this work. It is given by the ∞-category of *synthetic* spectra based on an Adams type homology theory E (seen as a commutative algebra in spectra). This construction was developed by [Pst18], with the goal of understanding Adams towers of spectra, and has since found a variety of applications such as Burklund–Hahn–Senger's proof of a conjecture of Galatius and Randall-Williams in [BHS19]. In [BHS20], Burklund–Hahn–Senger illustrate how synthetic spectra fit into their framework of deformations of homotopy theories, spurring our discussion. We will begin by recalling the construction of synthetic spectra as in [Pst18], and end the section with the main result; namely a description of synthetic spectra as a deformation of spectra.

We will now recall the construction and some fundamental facts about synthetic spectra, based on the original paper [Pst18]. In this entire discussion, we will fix an Adams type homology theory E, upon which our synthetic spectra will be based. For a definition of the former, see [Pst18] Definition 3.14.

8.1 Construction and basic properties

Synthetic spectra are defined in such a way that they recover the information coming from spectra, while also encoding information about the E-homology of spectra. In fact, they admit a very simple description as an ∞-category of sheaves of spectra.

Recall that the ∞-category of spectra Sp is presentable, and that its compact objects are given precisely by the finite spectra Sp^{ω}. The latter admit an interpretation as spectra weakly equivalent to a finite CW spectrum. By the presentability of Sp, we therefore have an equivalence

$$\mathrm{Sp} \simeq \mathrm{Ind}(\mathrm{Sp}^{\omega}).$$

We now try to mimic this, while also incorporating information about the E-homology of spectra. For this, let us define a full subcategory of finite spectra given by finite E-projective spectra.

Definition 8.1. Let Sp_E^{ω} be the full subcategory of Sp^{ω} on finite spectra X such that their E-homology E_*X is a finite projective E_*-module.

Note that this ∞-category is not sufficiently nice for doing homotopy theory. Indeed, it is pre-additive, since it is closed under finite direct sums and products in $\mathrm{Sp}^{\omega} \subset \mathrm{Sp}$, but it lacks more colimits. Finally, note that if

$$X \to Y \to Z$$

is a fibre sequence in Sp_E^{ω} such that the induced map $E_*Y \to E_*Z$ is a surjection, then the latter is split by projectivity of E_*Z. This is a desirable proprety, and we will enforce it to hold in the colimit completion as well. The desiderata above are quantified in the following definition.

Definition 8.2. Consider the following constructions on the ∞-category Sp_E^{ω}.

1. First, let us freely adjoint all colimits, i.e. form the presentable ∞-category $\mathcal{P}(\mathrm{Sp}_E^{\omega})$.
2. Second, let us preserve direct sums formed in Sp_E^{ω}, which amounts to restricting to product-preserving presheaves. These form a full subcategory denoted

   $$\mathcal{P}_{\Sigma}(\mathrm{Sp}_E^{\omega}) = \mathcal{F}\mathrm{un}^{\Pi}((\mathrm{Sp}_E^{\omega})^{\mathrm{op}}, \mathcal{S})$$

   and referred to as the ∞-category of spherical presheaves.
3. Third, let us recover fibre sequences in Sp_E^{ω} by considering the full subcategory on spherical presheaves F such that if

   $$X \to Y \to Z$$

   is a fibre sequence in Sp_E^{ω} with $E_*Y \to E_*Z$ a surjection, then

   $$F(Z) \to F(Y) \to F(X)$$

   is a fibre sequence.

Finally, let us stabilise this, to obtain a full subcategory

$$\mathcal{S}\mathrm{yn}_E \subset \mathcal{F}\mathrm{un}^{\Pi}((\mathrm{Sp}_E^{\omega})^{\mathrm{op}}, \mathrm{Sp})$$

of synthetic spectra based on E.

This definition still requires some clarification. Indeed, the second and third conditions might not interact well, stabilisation might ruin this, and we do not know what kind of ∞-category $\mathcal{S}\mathrm{yn}_E$ is. The most efficient way to patch these compatibility questions is to phrase the third condition as a sheaf condition on our spherical presheaves with respect to a Grothendieck topology on Sp_E^{ω}. Since sheaves are well behaved as categorical objects, we will see that the spherical subcategory and the stabilisation procedure interact well.

Definition 8.3. Consider the Grothendieck pretopology on Sp_E^{ω} in which a a map $Y \to Z$ is a covering if and only if the induced map

$$E_*Y \to E_*Z$$

is surjective.

In fact, this Grothendieck pretopology behaves well with respect to the preadditivity of Sp_E^{ω} as well as its symmetric monoidal structure obtained as the restriction of the smash product of spectra. This makes the resulting site into an excellent ∞-site in the sense of [Pst18] Definition 2.24. Let us first note that this recovers the third condition precisely as required.

Remark 8.1. Equipping Sp_E^{ω} with the Grothendieck pretopology as above, we see that the spherical sheaves on this site are precisely those that satisfy the third condition in Definition 8.2. This stated more generally and well documented in [Pst18] Theorem 2.8.

Proof. Let $F \in \mathcal{S}\mathrm{hv}(\mathrm{Sp}_E^{\omega})$ be a spherical sheaf on the site of finite E-projective spectra. We need to show that if $X \to Y \to Z$ is a fibre sequence in the underlying site such that the induced map $E_*Y \to E_*Z$ is a surjection, then $F(Z)$ is the fibre of $F(Y) \to F(X)$. Note that per definition, this means that $Y \to Z$ is a covering in the underlying site.

Now we use the fact that F is a sheaf to note that its value on Z can be obtained as the limit of the diagram obtained by applying F to the Čech nerve of the covering $Y \to Z$. In fact, let us choose a covering of every term in the fibre sequence. This results in a diagram of the form

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \times_Z Y \arrow[r] \arrow[d] & Y \arrow[d] \\
X \arrow[r] & Y \arrow[r] & Z.
\end{tikzcd}$$

Note that the rightmost arrow is a cover by assumption, while the centre arrows are obtained as refinements and trivial covers, hence are also covers by the axioms of a Grothendieck topology. Now note that F was assumed to be spherical, i.e. preserve products. We can therefore consider the usual Čech resolutions associated to every cover in this diagram to obtain a cosimplicial diagram of short sequences

$$\begin{tikzcd}
F(Y^{\times_Z \bullet}) \arrow[r] & F((Y \times_Z Y)^{\times_Y \bullet}) \arrow[r] & F(X^{\times_X \bullet}) \\
F(Y \times_Z Y) \arrow[r] \arrow[u] \arrow[u, shift left] \arrow[u, shift right] & F((Y \times_Z Y) \times_Y (Y \times_Z Y)) \arrow[r] \arrow[u] \arrow[u, shift left] \arrow[u, shift right] & F(X \times_X X) \arrow[u] \arrow[u, shift left] \arrow[u, shift right] \\
F(Y) \arrow[r] \arrow[u, shift left] \arrow[u, shift right] & F(Y \times_Z Y) \arrow[r] \arrow[u, shift left] \arrow[u, shift right] & F(X) \arrow[u, shift left] \arrow[u, shift right] \\
F(Z) \arrow[r] \arrow[u] & F(Y) \arrow[r] \arrow[u] & F(X).
\end{tikzcd}$$

In this diagram the horizontal arrows are induced by the original fibre sequence, while the cosimplicial structure maps in the vertical directions are induced by projections. Now we use that fact that F was a sheaf to see that all three vertical augmented cosimplicial diagrams are limit diagrams by descent. Further, note that in cosimplicial degree $\bullet \geq 0$ (with the cosimplicial degree $\bullet = -1$ being the augmentation), all horizontal sequences are fibre sequences. Indeed, we see that this consists of F applied to the fibre sequences

$$X^{\times_X \bullet+1} \to (Y \times_Z Y)^{\times_Y \bullet+1} \to Y^{\times_Z \bullet+1}.$$

Now the latter admits a section, precisely given by the product of (relative) diagonal morphisms

$$\Delta^{\bullet+1}_{Y\to Z} : Y^{\times_Z \bullet+1} \to (Y \times_Z Y)^{\times_Y \bullet+1}.$$

This means that the original fibre sequence is split, and one can express the middle term as a direct sum of the two outside terms. Now it is clear that a spherical presheaf sends a split fibre sequence to a (split) fibre sequence precisely since it preserves this product. We conclude that in all nonzero cosimplicial degrees, the horizontal sequences are fibre sequences. By the sheaf property of F, we further know that the final sequence on augmentations, in cosimplicial degree -1, is the limit of the sequences in cosimplicial degree ≥ 0. To finish the proof, we then simply commute the limit coming from the fibre sequences with the cosimplicial limit in the vertical direction, to obtain that the bottom sequence

$$F(Z) \to F(Y) \to F(X)$$

is a limit of limit diagrams, ergo a limit diagram itself.

Conversely, let F be a spherical presheaf with the special property that it sends a fibre sequence of a covering map to a fibre sequence of spaces. To show that it is a sheaf, we use a similar proof strategy as before, where we let $Y \to Z$ be an arbitrary covering with fibre X, and consider the map of fibre sequences

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[d] \\ 0 \arrow[r] & Z \arrow[r] & Z. \end{tikzcd}$$

In this diagram, all vertical maps are covering maps simply by the axioms of a Grothendieck topology. Once again, we take Čech nerves of all these coverings and apply F to obtain a sequence of cosimplicial diagrams

$$\begin{tikzcd} F(Z^{\times_Z \bullet}) \arrow[r] & F(Y^{\times_Z \bullet}) \arrow[r] & F(X^{\times_0 \bullet}) \\ F(Z \times_Z Z) \arrow[r] \arrow[u, shift left=2] \arrow[u] \arrow[u, shift right=2] & F(Y \times_Z Y) \arrow[r] \arrow[u, shift left=2] \arrow[u] \arrow[u, shift right=2] & F(X \times_0 X) \arrow[u, shift left=2] \arrow[u] \arrow[u, shift right=2] \\ F(Z) \arrow[r] \arrow[u, shift left] \arrow[u, shift right] & F(Y) \arrow[r] \arrow[u, shift left] \arrow[u, shift right] & F(X) \arrow[u, shift left] \arrow[u, shift right] \\ F(Z) \arrow[r] \arrow[u] & F(Z) \arrow[r] \arrow[u] & F(0). \arrow[u] \end{tikzcd}$$

Now note that at every cosimplicial level $\bullet \geq 0$, the horizontal sequences are fibre sequences. Indeed, before applying F, we see that

$$Z^{\times_Z \bullet+1} \to Y^{\times_Z \bullet+1} \to X^{\times \bullet+1}$$

is a fibre sequences. By assumption for $\bullet = 0$, and due to the fact that fibre sequences are invariant under pullbacks for higher cosimplicial degrees. If we now take the limit of these cosimplicial diagrams, it is clear that this will preserve the fibre sequences as well, whence we obtain another diagram of fibre sequences. Note that the bottom row is trivially a fibre sequence since a spherical presheaf must send 0 to 0, and the map on $F(Z)$ is the identity. We obtain the following diagram of fibre sequences:

$$\begin{tikzcd} \lim_\Delta F(Z^{\times_Z \bullet}) \arrow[r] \arrow[d] & \lim_\Delta F(Y^{\times_Z \bullet}) \arrow[r] \arrow[d] & \lim_\Delta F(X^{\times \bullet}) \arrow[d] \\ F(Z) \arrow[r] & F(Z) \arrow[r] & F(0). \end{tikzcd}$$

It is immediately obvious that the leftmost map is an equivalence, since the original cosimplicial diagram was constant at value $F(Z)$, and in fact it is not hard to show that the rightmost vertical map is an equivalence as well, since this is a particularly simple cosimplicial object. At this point, we would then like to apply the spectral five lemma discussed in the proof of Proposition 6.4. However, we are only working in the context of spaces, so that one can not necessarily obtain a long exact sequence of abelian groups by applying π_*. However, since Sp_E^ω is additive–meaning that every object admits the structure of a group-like $\mathbb{E}_\infty$-monoid with respect to the direct sum–and spherical presheaves preserve these direct sums, we may view F as landing in group-like $\mathbb{E}_\infty$-spaces, which are none other than connective spectra. Therefore, one can view this fibre sequence as lying in $\mathrm{Sp}_{\geq 0}$, whence one can apply the five lemma, and it suffices to show that the right vertical map is an equivalence.

The identification of $\lim_\Delta F(X^{\times\bullet})$follows from general consideration of cosimplicial objects associated to iterated products. Indeed, sphericity of F guarantees that this cosimplicial diagram is of the form $F(X)^{\oplus\bullet}$. Now note that any cosimplicial spectrum (where we once again view these presheaves as landing in connective spectra) admits a spectral sequence called the Bousfield–Kan spectral sequence that converges to the homotopy groups of its limit. In general, the Bousfield–Kan spectral sequence induced by applying a sheaf to the Čch nerve of a covering recovers the descent spectral sequence, but in the case of the cover $X \to 0$ it becomes rather trivial. Indeed, we claim that the limit of the cosimplicial object $F(X)^{\oplus\bullet}$ is clearly seen to be 0 by a closed look at the Bousfield–Kan spectral sequence. For the sake of brevity of this proof, we outsource the discussion of this result to Lemma 8.1 below. □

Remark 8.2. Let $C^\bullet$ be a cosimplicial spectrum, then there is a spectral sequence called the Bousfield–Kan spectral sequence with of the form

$$E_2^{p,q} = \pi^p(\pi_q C^\bullet) \implies \pi_{q-p} \lim_\Delta C^\bullet.$$

In this notation, the p-th cohomotopy group π^p of the cosimplicial abelian group $\pi_q C^\bullet$ refers to the p-th cohomology group of the cochain complex obtained from $\pi_q C^\bullet$ by the Dold–Kan equivalence for abelian groups. The existence and construction of the Bousfield–Kan spectral sequence is an entirely classical result from homological algebra, and we refer to [Gui07] for a description.

Lemma 8.1. *If the cosimplicial spectrum $C^\bullet$ above is of the form*

$$C^\bullet = T^{\oplus\bullet}$$

for some fixed spectrum T, with cofaces and codegeneracies given by the obvious projections and inclusions, then the cochain complex obtained by applying Dold–Kan to $\pi_q C^\bullet$ is acyclic.

Proof. Instead of analysing the entire cochain complex, we will consider a quasi-isomorphic cochain complex obtained as the normalisation of the former. This construction and the fact that it is quasi-isomorphic to the usual cochain complex obtained from the Dold–Kan equivalence is outlined in [Gui07]. This tells us that we can consider the cochain $N^*\pi_q C^\bullet$ complex whose m-th degree is given by

$$N^m \pi_q C^\bullet = \bigcap_{j=0}^{m-1} \ker(\pi_q C^m \xrightarrow{s^j} \pi_q C^{m-1}).$$

Now using our assumption on $C^\bullet$, we can rewrite this codegeneracy as simply a projection killing the j-th copy of $\pi_q T$

$$s^j : (\pi_q T)^{\oplus m} \to (\pi_q T)^{\oplus m-1},$$

whose kernel is then precisely the j-th copy of $\pi_q T$. Taking the intersection over all values of j, we see that each projection kills a different factor, so that the intersection is zero. We conclude that the normalised cochain complex obtained from the cosimplicial object $\pi_q C^\bullet$ is actually zero, whence its cohomology groups, i.e. the cohomotopy groups oft the cosimplicial abelian groups $\pi_q C^\bullet$ are trivial. □

It is then clear from this result that the E_2-page of such a case of the Bousfield–Kan spectral sequence–ergo also the homotopy groups to which it converges–are trivial.

We conclude from this proof that the sheaf condition is well understood, and recovers precisely what we need. The fact that sheafification interacts well with spherical (pre)sheaves will not be shown here, but we do refer to [Pst18] Section 2 for a complete discussion on why the sheafification functor sends spherical presheaves to spherical sheaves, making the latter into a left exact accessible localisation of the already stable presentable ∞-category $\mathcal{P}_\Sigma(\mathrm{Sp}_E^\omega)$, as well as a discussion of why one can do the same for hypersheaves and hypersheafification.

Now the final step is just to sheafify, so that one recovers the original definition

$$\mathcal{S}\mathrm{yn}_E = \mathcal{S}\mathrm{hv}_\Sigma^{\mathrm{Sp}}(\mathrm{Sp}_E^\omega) = \mathrm{Sp}(\mathcal{S}\mathrm{hv}_\Sigma(\mathrm{Sp}_E^\omega)).$$

The final identification is standard in the theory of spectrum-valued (spherical) sheaves, simply by noting that (spherical) sheaves are closed under limits as a full subcategory of (spherical) presheaves, so that the stabilisation procedure respects this, and one can view a stabilised (spherical) sheaf as a (spherical) sheaf of spectra.

This final approach allows us to define a number of functors in and out of synthetic spectra that will play an important role in the deformation picture. Several of these involve some sort of Yoneda embedding construction, so it will be useful to know that the Grothendieck site Sp_E^ω is subcanonical, i.e. such that representable presheaves are sheaves.

Lemma 8.2. *The site* Sp_E^ω *is subcanonical.*

Proof. Fixing some element $P \in \mathrm{Sp}_E^\omega$, we want to show that

$$\mathrm{map}(-, E) : \mathrm{Sp}_E^\omega \to \mathcal{P}(\mathrm{Sp}_E^\omega)$$

defines a (spherical) sheaf. Now it is clear that the resulting presheaf is spherical, whence one can use the characterisation of spherical sheaves established above. For this, let

$$X \to Y \to Z$$

be a fibre sequence of finite E-projective spectra, with $Y \to Z$ an E_*-surjection. We want to show that the induced sequence

$$\mathrm{map}(Z, P) \to \mathrm{map}(Y, P) \to \mathrm{map}(X, P)$$

is a fibre sequence of spaces. This is immediate simply by noting that Sp_E^ω is a full subcategory of the stable ∞-category Sp and closed under fibres and cofibres. It then suffices to apply the contravariant functor $\mathrm{map}(-, P)$ to obtain a fibre sequence. Further, note that this holds for P any spectrum. □

Definition 8.4. Using the characterisation

$$\mathcal{S}\mathrm{yn}_E \simeq \mathcal{S}\mathrm{hv}^{\mathrm{Sp}}(\mathrm{Sp}_E^\omega),$$

we can define the following functors:

- Consider the Yoneda embedding

  $$よ : \mathrm{Sp} \to \mathcal{P}(\mathrm{Sp}),$$

  whose image restricts to spherical presheaves $\mathcal{P}_\Sigma(\mathrm{Sp})$, by the remark in the proof of Lemma 8.2. By the same proof, one sees that further restricting to spherical presheaves on finite E-projective spectra actually lands in the full subcategory of spherical sheaves. We conclude that there exists a functor

  $$よ : \mathrm{Sp} \to \mathcal{P}_\Sigma(\mathrm{Sp}) \to \mathcal{S}\mathrm{hv}_\Sigma(\mathrm{Sp}_E^\omega).$$

  Note that this is a right adjoint to the natural functor

  $$\mathcal{S}\mathrm{hv}_\Sigma(\mathrm{Sp}_E^\omega) \to \mathcal{P}(\mathrm{Sp}_E^\omega) \to \mathrm{Sp}$$

induced by the inclusion and the natural cocontinuous functor obtained as the colimit extension of the inclusion of finite E-projective spectra into all spectra. Finally, one can stabilise, to obtain a composite

$$\nu : \mathrm{Sp} \xrightarrow{\text{よ}} \mathcal{S}\mathrm{hv}_\Sigma(\mathrm{Sp}_E^\omega) \xrightarrow{\Sigma^\infty_+} \mathrm{Sp}(\mathcal{S}\mathrm{hv}_\Sigma(\mathrm{Sp}_E^\omega)) \simeq \mathcal{S}\mathrm{yn}_E.$$

This functor is called the synthetic analogue. A lot can be said about this functor and its failure to preserve all colimits, or be strict symmetric monoidal. In fact, this will control the structure of a deformation on synthetic spectra.

- Similarly, one can define the spectral Yoneda embedding in terms of the spectral enrichment of a stable ∞-category. Mapping spectra will be denoted Map, and they allow us to define a functor

  $$Y : \mathrm{Sp} \to \mathcal{S}\mathrm{hv}_\Sigma^{\mathrm{Sp}}(\mathrm{Sp}_E^\omega)$$

  using the formula

  $$Y(X)(P) := \mathrm{Map}(P, X).$$

  Note that that this defines a spherical sheaf with values in spectra since the spectral enrichment once again sends colimits in the first variable to limits in the second, whence it is a spherical sheaf. To relate this to the synthetic analogue, note that

  $$\Omega^\infty \mathrm{Map}(P, X) \simeq \mathrm{map}(P, X).$$

  Indeed, since the site of finite E-projective spectra is additive, the Yoneda embedding takes values in sheaves of grouplike $\mathbb{E}_\infty$-monoids, and these can be identified using the suspension spectrum functor with the connective parts of mapping spectra, i.e.

  $$\nu X(P) = Y(X)(P)_{\geq 0}.$$

Some properties of the synthetic analogue are rather immediate from the definition.

Lemma 8.3. *The synthetic analogue is lax symmetric monoidal and commutes with filtered colimits.*

Proof. Note that ν sends a spectrum X to the levelwise suspension spectrum of the representable sheaf associated to X. Since this representable sheaf is to be seen as a sheaf on Sp_E^ω, and every object in the latter is finite ergo compact as an object of Sp, it is clear that mapping out of these preserves filtered colimits, and postcomposition with the left adjoint Σ^∞_+ will preserve these as well. To see that ν is lax symmetric monoidal, it suffices to note that the Yoneda embedding was right adjoint to the embedding of Sp_E^ω into Sp. Now this embedding is symmetric monoidal since both sides are equipped with the smash product, whence its colimit extension is symmetric monoidal as well by the Day convolution structure on presheaves. We infer that its left adjoint must be lax symmetric monoidal. Note that the formation of suspension spectra is symmetric monoidal, so that the composite ν of the suspension spectrum functor and the Yoneda embedding is lax symmetric monoidal. □

Lemma 8.4. *The presentable stable ∞-category $\mathcal{S}\mathrm{yn}_E$ is generated under colimits by objects of the form νP, where P ranges over all finite E-projective spectra.*

Proof. Note that due to Yoneda's lemma, the synthetic analogue has a universal property. If P is a finite E-projective spectrum and X any synthetic spectrum, we obtain

$$\begin{aligned} \mathrm{map}(\nu P, X) &\simeq \mathrm{map}(\Sigma^\infty_+ \text{よ}(P), X), \\ &\simeq \mathrm{map}(\text{よ}(P), \Omega^\infty X), \\ &\simeq \Omega^\infty X(P). \end{aligned}$$

Therefore, it is clear that the family $\{\mathrm{map}(\Sigma^i \nu P, -)\}$ is conservative, where i ranges over the integers and P ranges over all finite E-projective spectra. We conclude that the synthetic spectra νP generate $\mathcal{S}\mathrm{yn}_E$. □

Lemma 8.5. *The synthetic analogue ν is strict symmetric monoidal when one of the inputs is E, in the sense that for any spectrum X the comparison map*

$$\nu E \otimes \nu X \to \nu(E \wedge X)$$

is an equivalence.

Proof. Recall from the proof of Lemma 8.4 that it suffices to show that in this case the comparison map on representable presheaves is an equivalence, since the suspension spectrum functor is strict symmetric monoidal. More precisely, we have a comparison map of spherical sheaves of spaces

$$\text{よ}(E) \otimes \text{よ}(X) \to \text{よ}(E \wedge X),$$

which on a finite E-projective spectrum P evaluates to a map of spaces

$$(\text{よ}(E) \otimes \text{よ}(X))(P) \to \text{よ}(E \wedge X)(P) \simeq \mathrm{map}(P, E \wedge X).$$

Now the left hand side can also be identified explicitly. Indeed, we are considering a Day convolution product of presheaves, but the latter has the universal property that it makes the Yoneda embedding of Sp_E^ω into its ∞-category of presheaves symmetric monoidal. More explicitly, this means that for some finite E-projective Q, the functor

$$\text{よ}(Q) \otimes - : \mathcal{P}(\mathrm{Sp}_E^\omega) \to \mathcal{P}(\mathrm{Sp}_E^\omega)$$

is the left Kan extension of

$$Q \wedge - : \mathrm{Sp}_E^\omega \to \mathrm{Sp}_E^\omega \xrightarrow{\text{よ}} \mathcal{P}(\mathrm{Sp}_E^\omega)$$

along the Yoneda embedding よ, giving rise to a commutative diagram

$$\begin{tikzcd}
\mathrm{Sp}_E^\omega \arrow[r, "Q\wedge -"] \arrow[d, "\text{よ}"] & \mathrm{Sp}_E^\omega \arrow[r, "\text{よ}"] & \mathcal{P}(\mathrm{Sp}_E^\omega), \\
\mathcal{P}(\mathrm{Sp}_E^\omega) \arrow[urr, "\text{よ}(Q)\otimes -"'] & &
\end{tikzcd}$$

Further, let us use that E is Adams type, hence can be written as a filtered colimit

$$E \simeq \operatorname*{colim}_\alpha E_\alpha, \; E_\alpha \in \mathrm{Sp}_E^\omega.$$

We come to the description

$$\begin{aligned}
(\text{よ}(E) \otimes \text{よ}(X))(P) &\simeq (\text{よ}(\operatorname*{colim}_\alpha E_\alpha) \otimes \text{よ}(X))(P), \\
&\simeq \operatorname*{colim}_\alpha (\text{よ}(E_\alpha) \otimes \text{よ}(X))(P), \\
&\simeq \operatorname*{colim}_\alpha \text{よ}(E_\alpha \wedge X)(P), \\
&\simeq \operatorname*{colim}_\alpha \mathrm{map}(P, E_\alpha \wedge X) \\
&\simeq \mathrm{map}(P, E \wedge X).
\end{aligned}$$

In this identification, we simultaneously used that the Yoneda embedding commutes with filtered colimits by compactness of finite E-projective spectra, as well as the observation that the Day convolution commutes with colimits in each variable. The rest of the proof is then a simple matter of manipulating the expressions using the commutative diagram established above. □

8.2 Bigraded spheres and τ

Now we can begin to describe synthetic spectra as a generalised homotopy theory, defining a notion of spheres and homotopy groups. In fact, the homotopy groups of synthetic spectra are naturally bigraded, where one grading reflects that fact that the sheaves that they are defined as take values in spectra, and the second grading encapsulates the internal homotopy theory of the ∞-category of finite E-projective spectra. In fact, these bigraded synthetic homotopy groups interpolate between the homotopy groups of spectra and their E_*-homology as will be shown later on.

Definition 8.5. For any integers t, w referred to as the topological degree and the weight respectively, let us define a synthetic spectrum by

$$\mathbb{S}^{t,w} := \Sigma^{t-w} \nu \mathbb{S}^{w},$$

where we have taken the pointwise suspension of the sheaf obtained as the synthetic analogue of the shifted sphere spectrum $\mathbb{S}^{w}$. The integer $t - w$ is called the Chow degree.

Having defined these bigraded spheres, one can then define bigraded homotopy groups of a synthetic spectrum X by

$$\pi_{t,w} X := \pi_0 \mathrm{map}(\mathbb{S}^{t,w}, X).$$

Remark 8.3. Note from the definition that

$$\Sigma \mathbb{S}^{t,w} \simeq \Sigma^{t-w+1} \nu \mathbb{S}^{w} \simeq \Sigma^{(t+1)-w} \nu \mathbb{S}^{w} = \mathbb{S}^{t+1,w}.$$

This means on mapping spaces we have

$$\mathrm{map}(\mathbb{S}^{t+1,w}) \simeq \mathrm{map}(\Sigma \mathbb{S}^{t,w}, X) \simeq \mathrm{map}(\mathbb{S}^{t,w}, \Omega X) \simeq \Omega \mathrm{map}(\mathbb{S}^{t,w}, X),$$

whence letting t vary, these spaces define a spectrum object in spaces. In particular, this means that our formula for bigraded synthetic homotopy groups can be expressed in terms of mapping spectra in the stable ∞-category $\mathcal{S}\mathrm{yn}_E$, where we now let the topological weight t be encapsulated in the spectrum, and define

$$\pi_{t,w} X = \pi_t \mathrm{Map}(\mathbb{S}^{0,w}, X)$$

for any integers t, w. We will see later that this has a computational upshot: all bigraded synthetic homotopy groups can be recovered simply by mapping out of the spheres $\mathbb{S}^{0,w}$ in an appropriately enriched way. In fact, this is an essential result for the deformation picture of synthetic spectra.

Remark 8.4. More generally, we have a product formula for bigraded spheres of the form

$$\mathbb{S}^{t,w} \otimes \mathbb{S}^{t',w'} \simeq \mathbb{S}^{t+t',w+w'}.$$

This is an immediate consequence of the definition and the key observation that ν is strict symmetric monoidal when restricted to the full subcategory of finite E-projective spectra (containing all spheres). Indeed, in that case ν is simply obtained as the composite of the symmetric monoidal functor Σ^{∞}_{+} on sheaves and the Yoneda embedding of Sp_E^{ω}–sans restriction–which is strict symmetric monoidal by construction of the Day convolution (cf. the proof of Lemma 8.5). Explicitly, this tells us that

$$\nu \mathbb{S}^{w} \otimes \nu \mathbb{S}^{w'} \simeq \nu(\mathbb{S}^{w} \wedge \mathbb{S}^{w'}) \simeq \nu \mathbb{S}^{w+w'}$$

whence the observation above follows immediately.

Recall that ν did not preserve cofibre sequences, and we claimed that this was key to the deformation structure of synthetic spectra. In fact, synthetic spectra admit an internal description of the deformation parameter τ in terms of precisely this deficit. If we let X be a synthetic spectrum and P a finite E-projective spectrum, then one can consider the cofibre ΣP of the map $P \to 0$. Note that the functor X sends this diagram to

$$X(P) \to X(0) \simeq 0,$$

with the last equivalence following from sphericity of X. The fibre of this induced map is $\Omega X(P)$, with the desuspension being constructed levelwise. Now this is not necessarily equivalent to $X(\Sigma P)$, but there is an entirely formal comparison map (natural in P)

$$\tau : X(\Sigma P) \to \Omega X(P)$$

induced by the fact that the natural map $X(\Sigma P) \to X(P)$ is zero, hence factors through the fibre. We are primarily interested in the special case where $X = \mathbb{S}^{0,0}$, so that this comparison map takes the form

$$\tau : \nu\mathbb{S}(\Sigma P) \to \Omega\nu\mathbb{S}(P).$$

Now the right hand side is given by

$$\nu\mathbb{S}(\Sigma P) \simeq \Sigma^\infty_+ \mathrm{map}(\Sigma P, \mathbb{S}) \simeq \Sigma^\infty_+ \mathrm{map}(P, \Omega\mathbb{S}) \simeq \Sigma^\infty_+ \mathrm{map}(P, \mathbb{S}^{-1}) \simeq \nu\mathbb{S}^{-1}(P) \simeq \Omega\mathbb{S}^{-1,-1}(P)$$

so that the comparison map can be seen as a natural transformation i.e. morphism of synthetic spectra

$$\tau : \mathbb{S}^{-1,-1} \simeq \nu\mathbb{S}^{-1} \to \Omega\mathbb{S}^{0,0} \simeq \mathbb{S}^{-1,0}.$$

Of course, it is customary to identify τ with its suspensions, and we will usually work with τ viewed as a map

$$\tau : \mathbb{S}^{0,-1} \to \mathbb{S}^{0,0},$$

in particular defining a class in degree $(0, -1)$ in the synthetic stable stem. It is then a result of [Pst18] that this comparison map for all synthetic spectra is obtained from the one above by tensoring with the identity on a given synthetic spectrum. We refer to Proposition 4.18 in op. cit. for a proof[9].

Just as in any deformation, the parameter τ plays a fundamental role. We denote its cofibre by $C\tau$, sitting in the cofibre sequence

$$\mathbb{S}^{0,-1} \to \mathbb{S}^{0,0} \to C\tau.$$

In fact, this element controls the t-structure on synthetic spectra. It is not only a synthetic spectrum, but actually an $\mathbb{E}_\infty$-algebra in synthetic spectra. This is nontrivial a priori, but will of course become trivial once we identify it with the image of the $\mathbb{E}_\infty$-algebra $c\tau$ in filtered spectra under a symmetric monoidal left adjoint exhibiting $\mathcal{S}\mathrm{yn}_E$ as a deformation. Without this information, one can show that $C\tau$ is an $\mathbb{E}_\infty$-algebra manually, by showing that it is the image of the $\mathbb{E}_\infty$-algebra $\mathbb{S}^{0,0}$ under the lax monoidal functor $\tau_{\leq 0}$ arising in the t-structure on synthetic spectra. The following section is devoted to showing this result.

8.2.1 The t-structure

Note that $\mathcal{S}\mathrm{yn}_E$ could be expressed as a stable ∞-category of spherical sheaves of spectra. It therefore obtains a natural t-structure in terms of sheaves of homotopy groups. Given some synthetic spectrum X, one obtains for every n a (spherical) presheaf of abelian groups

$$\pi_n X^{\mathrm{pre}} : P \mapsto \pi_n X(P).$$

This is not necessarily a sheaf, so it must be sheafified to obtain the sheaf homotopy groups $\pi_n X$. It is in terms of these sheaf homotopy groups that one makes the usual identification of the connective part with the full subcategory on X such that $\pi_n X$ vanishes for $n < 0$. Note that the sheafification can change this presheaf of abelian groups quite a lot, so that the natural t-structure on synthetic spectra is not a levelwise t-structure on some diagram category in spectra.

Since this t-structure is defined in a rather formal way, it is easy to see that it is (relatively) well behaved: it is compatible with filtered colimits and the symmetric monoidal structure, and is right complete, as in [Pst18] Proposition 2.16. In this section, we aim to highlight its relation to the thread operator τ.

[9] In fact, this observation means that synthetic spectra on which τ acts invertibly are precisely those synthetic spectra that send suspensions of finite E-projective spectra to loops. This description of the generic fibre is key to the description of deformations of unstable homotopy theories using algebraic theories as described in [Bal21].

Remark 8.5. In many contexts, it can be more insightful to consider an additive presentable ∞-category, since its stabilisation admits a natural t-structure in which the original additive ∞-category forms the subcategory of connective objects. For example, the ∞-category of spectra is useless without its natural t-structure, so that it makes more sense to consider simply the additive presentable ∞-category of group-like $\mathbb{E}_\infty$ -monoids in spaces, considered as the connective part in its stabilisation–namely Sp. Implicit in this consideration is that there is a factorisation

$$\mathcal{P}r^{\mathrm{L}} \to \mathcal{P}r^{\mathrm{L}}_{*} \to \mathcal{P}r^{\mathrm{L}}_{\mathrm{add}} \to \mathcal{P}r^{\mathrm{L}}_{\mathrm{St}}$$

of the stabilisation functor through universal pointed and additive presentable ∞ -categories. This was hinted at in Section 4.3, and is explained in detail in [GGN15].

However, this perspective is not what gives rise to the t-structure on synthetic spectra. Indeed, the ∞-category of spherical sheaves of spaces on Sp_E^{ω} is not assumed to be hypercomplete, so that homotopy presheaves need to be sheafified, and many operations are not done levelwise. In [PV19] the authors instead consider the subcategory of hypercomplete spherical sheaves, and its sheafification: the ∞-category of hypercompletete spherical sheaves of spectra. As is a common theme, hypercomplete sheaves give rise to a coherent ∞-topos in which the t-structure is well behaved, and this is desirable if one wants to set up e.g. a form of obstruction theory inside synthetic spectra. In our case, we work with a messier t-structure in which the connective part does not admit a direct identification, and such that it admits ∞-connective objects. In fact, this will be essential to the recollement picture.

The first result is that the τ-cofibre sequence actually resembles a decomposition into coconnective and 1-connective parts.

Proposition 8.1. *Let X be a spectrum, then there is a cofibre sequence*

$$\Sigma^{0,-1}\nu X \to \nu X \to C\tau \otimes \nu X.$$

This is such that one can identify

$$\Sigma^{0,-1}\nu X \simeq \tau_{\geq 1}\nu X, \qquad \nu X \simeq \tau_{\geq 0}\nu X, \qquad C\tau \otimes \nu X \simeq \tau_{\leq 0}\nu X.$$

Proof. First, note that νX is 0-connective per construction. We conclude that the middle equivalence holds. As for the left equivalence, it is clear that $\Sigma^{0,-1}\nu X$ is given by

$$\Sigma^{0,-1}\nu X = \Sigma\nu\mathbb{S}^{-1} \otimes \nu X = \Sigma\nu(\Sigma^{-1}X)$$

since it is the source of the colimit-to-limit comparison map τ. We then note that this is given by the suspension of a synthetic analogue, hence is $(0 + 1 = 1)$-connective. Finally, it suffices to show that the cofibre $C\tau \otimes \nu X$ is 0-coconnective. Here let us be careful, since the colimit-to-limit comparison map was actually defined to be the shift of τ, so that showing that $C\tau \otimes \nu X$ is 0-coconnective amounts to showing that

$$\mathrm{cof}((\nu\Omega X)(P) \to \Omega\nu X(P))$$

is -1-coconnective. Indeed, since sheafification is exact and n-coconnective objects in $\mathcal{S}\mathrm{yn}_E$ are sheaves of n-coconnective spectra, so it suffices to show this levelwise. We can rewrite this cofibre as

$$\mathrm{cof}(\Sigma^\infty\Omega\mathrm{map}(P, X) \to \Omega\Sigma^\infty\mathrm{map}(P, X)).$$

Now Ω commutes with Σ for spectra since they are mutually inverse, so that this is almost true for the connective spectra above, modulo the information in degree -1 introduced by the Ω in the target. We conclude that the cofibre is (-1)-coconnective, and in fact concentrated in degree -1. ◻

Remark 8.6. If $X = \mathbb{S}$, we see that $C\tau \simeq \tau_{\leq 0}\mathbb{S}^{0,0}$, so that it obtains a natural structure of an $\mathbb{E}_\infty$-algebra in $\mathcal{S}\mathrm{yn}_E$. Indeed, we stated earlier that the t-structure was compatible with the symmetric monoidal structure, so that the Postnikov truncation functors are lax symmetric monoidal.

Corollary 8.1. *From the description above, we see that*

$$\tau_{\geq 1}\nu X \simeq \mathrm{fib}(\nu X \to C\tau \otimes \nu X).$$

Now $\tau_{\geq 1}\nu X$ is (the suspension of) the synthetic analogue of $\Sigma^{-1}X$, so that this process can be iterated, leading to

$$\tau_{\geq 2}\nu X \simeq \mathrm{fib}(\Sigma^{0,-1}X \to C\tau \otimes \Sigma^{0,-1}\nu X),$$

etc.

In fact, this can be done for all $n \geq 0$ to obtain a tower

$$\begin{tikzcd}
\cdots \arrow[r] & \tau_{\geq n+1}\nu X \arrow[r] & \tau_{\geq n}\nu X \arrow[r] \arrow[d] & \cdots \arrow[r] & \tau_{\geq 2}\nu X \arrow[r] & \tau_{\geq 1}\nu X \arrow[r] \arrow[d] & \tau_{\geq 0}\nu X \arrow[d] \\
& & C\tau \otimes \tau_{\geq n}\nu X & & & C\tau \otimes \tau_{\geq 1}\nu X & C\tau \otimes \nu X
\end{tikzcd}$$

where the vertical arrows indicate cofibre sequences, i.e. the next level in the tower is obtained as the fibre of the unit map leaving from the previous one into its tensor product with $C\tau$. We will see in Section 8.3 that this is precisely the $C\tau$-Adams tower of νX. We can therefore say that the Postnikov tower of a synthetic analogue is precisely its $C\tau$-Adams tower, thus reifying the intuition that τ controls the t-structure in $\mathcal{S}\mathrm{yn}_E$.

While the t-structure on $\mathcal{S}\mathrm{yn}_E$ is relatively well understood, as is clear from the discussion above, its relation to bigraded homotopy groups arising from the spheres $\mathbb{S}^{t,w}$ is nontrivial and involves information coming from the Adams type homology theory E underlying the construction. For example, the bigraded homotopy groups $\pi_{t,w}\nu X$ can a priori have information in negative Chow degrees. In positive Chow degrees however, we simply recover the homotopy groups of X.

Proposition 8.2 ([Pst18] Corollary 4.12)**.** *Let X be a spectrum, then in postive Chow degree, i.e. for $t - w \geq 0$, we have*

$$\pi_{t,w}\nu X \cong \pi_t X.$$

Proof. This follows immediately from the definition and the universal property of the synthetic analogue. Indeed, one computes

$$\begin{aligned}
\pi_{t,w}\nu X &\simeq \pi_0\mathrm{map}(\mathbb{S}^{t,w}, \nu X),\\
&\simeq \pi_0\mathrm{map}(\Sigma^{t-w}\nu\mathbb{S}^{w}, \nu X),\\
&\simeq \pi_0\mathrm{map}(\nu\mathbb{S}^{w}, \Omega^{t-w}\nu X),\\
&\simeq \pi_{t-w}\mathrm{map}(\nu\mathbb{S}^{w}, \nu X),\\
&\simeq \pi_{t-w}\mathrm{map}(\Sigma^{\infty}\text{よ}(\mathbb{S}^{w}), \nu X),\\
&\simeq \pi_{t-w}\mathrm{map}(\text{よ}(\mathbb{S}^{w}), \Omega^{\infty}\nu X),\\
&\simeq \pi_{t-w}\Omega^{\infty}\nu X(\mathbb{S}^{w}),\\
&\simeq \pi_{t-w}\mathrm{map}(\mathbb{S}^{w}, X),\\
&\simeq \pi_{t-w}\Omega^{w}\mathrm{map}(\mathbb{S}, X),\\
&\simeq \pi_t X.
\end{aligned}$$

□

While the negative Chow degree part might not be known explicitly, if the underlying spectrum has the structure of a homotopy E-module, more can be said: namely that the negative Chow degree groups vanish. The proof of this argument requires a careful analysis of the long exact sequences arising from these bigraded homotopy groups and their relation to the τ-inversion functor; but we elide the proof and simply propose the statement below.

Proposition 8.3 ([Pst18], 4.60). *Let M be a homotopy E-module in* Sp*. For our considerations, one can usually think of M as a smash product $M \simeq E \wedge X$. Then the bigraded homotopy groups*

$$\pi_{t,w} \nu M$$

vanish in negative Chow degree, i.e. for $t - w < 0$.

8.3 Complete objects and Adams towers

Just as regular homotopy theory allows us to understand E-nilpotent complete spectra entirely in terms of the E-based Adams spectral sequence coming from their E-based Adams tower, a similar operation can be done in synthetic spectra. Replacing a spectrum X by its synthetic analogue, and replacing E by νE, we can develop a theory of the νE-based Adams towers of νX, which is related to the E-based Adams tower of X by a shift encoded precisely in the operator τ. The main application of this νE-based Adams spectral sequences is in the work of Burklund–Hahn–Senger on manifold theory in [BHS19], and in fact this section is mainly an elaboration on appendix A in op. cit. in which these Adams towers are described explicitly.

Let us begin by recalling the construction of the classical Adams spectral sequence for spectra, based on E.

Definition 8.6. Let X be some spectrum, and consider the map $X \to X \wedge E$ induced by tensoring the identity of X with the unit map $\mathbb{S} \to E$ of the ring spectrum E. Now define a spectrum X_1 as the fibre along this map

$$X_1 \to X \to X \wedge E.$$

In fact, since Sp is presentable and stable, the smash product commutes with cofibres, so one might as well construct X_1 by smashing X with the fibre $\bar{E}$ of the unit map.

Now note that one can iterate this process, defining X_{n+1} to be the fibre of the unit map

$$X_{n+1} \to X_n \to X_n \wedge E.$$

The canonical maps $X_{n+1} \to X_n$ assemble to form a tower object in spectra (a functor from $\mathbb{Z}_{\geq 0}^{\mathrm{op}}$) of the form

$$\cdots \to X_2 \to X_1 \to X_0 \simeq X,$$

the associated graded in degree n is $X_n \wedge E$ per construction, and using the stability of spectra. This tower is called the E-based Adams tower of X and its associated spectral sequence is the E-based Adams spectral sequence of X.

Remark 8.7. Since Sp is presentably symmetric monoidal and stable, we see that all involved operations of smash products and (co)fibres actually commute with smashing with another spectrum Y. Therefore, we see that the E-based Adams tower for $X \wedge Y$ is nothing else than the E-based Adams tower for X smashed with Y in every degree. In particular, it often suffices to describe the E-based Adams tower of the monoidal unit $\mathbb{S}$.

Remark 8.8. The most important part of the Adams tower is that its associated spectral sequence converges to the homotopy groups of the E-nilpotent completion $X_E^{\wedge}$ of X, i.e. the limit of the cosimplicial object $X \wedge E^{\wedge \bullet}$ obtained by smashing X with the cobar resolution of E.

Note that the definition of the Adams tower and its associated spectral sequence really works in any stable presentably symmetric monoidal ∞-category, where E can now be replaced by any commutative algebra object[10] Now we would like to mimic this construction in synthetic spectra and compare it to the Adams tower of X. The νE-based Adams tower of νX is defined similarly:

- Set $(\nu X)_0 = \nu X$,

[10] In fact, we only require E to be an $\mathbb{E}_1$-algebra, but for our considerations we are primarily interested in commutative ring objects, and actually only require them to be homotopy commutative.

- For a natural number n, define $(\nu X)_{n+1}$ to be the fibre of the unit map
$$(\nu X)_{n+1} \to (\nu X)_n \to (\nu X)_n \otimes \nu E.$$
- Iterate this to obtain a tower of synthetic spectra
$$\cdots \to (\nu X)_2 \to (\nu X)_1 \to (\nu X)_0 \simeq \nu X,$$
whose associated graded at degree n is $(\nu X)_n \otimes \nu E$.

In particular, in degree zero we see that we are considering the unit map
$$\nu X \to \nu X \otimes \nu E.$$
Now the target can actually be identified with $\nu(X \wedge E)$ by Lemma 8.5. Therefore, we are actually reduced to comparing
$$(\nu X)_1 = \mathrm{fib}(\nu X \to \nu(X \wedge E)), \qquad \nu X_1 = \nu\mathrm{fib}(X \to X \wedge E).$$
However, we know precisely what quantifies the difference between these two, namely the τ operator, inducing a comparison map
$$\nu X_1 \xrightarrow{\tau} (\nu X)_1.$$
Now by our previously cited observation that τ is merely induced from the map
$$\tau : \mathbb{S}^{0,0} \to \mathbb{S}^{0,1},$$
we conclude that there is an equivalence $(\nu X)_1 \simeq \Sigma^{0,1}\nu X_1$.

This tells us that the νE-based Adams tower of νX can be recovered from the E-based Adams tower of X by shifting in the deformation direction at every level. Indeed, replacing νX by $(\nu X)_n$ in the reasoning above, we can iterate to obtain a complete description
$$\cdots \to \Sigma^{0,2}\nu X_2 \to \Sigma^{0,1}\nu X_1 \to \nu X_0 \simeq \nu X$$
of the νE-based Adams tower of νX.

Remark 8.9. We will discuss in the next section that inverting τ is a left inverse to the synthetic analogue construction. We now see that even the νE-based Adams tower of νX only differs from (the synthetic analogue of) that of νX by repeated application of τ. Therefore, we see that the τ-inversion of the νE-based Adams tower of νX in synthetic spectra recovers the E-based Adams tower of X in ordinary spectra–or more properly, its image under the spectral Yoneda embedding exhibiting the equivalence of the latter with τ-invertible synthetic spectra. This important remark highlights the way in which synthetic homotopy theory recovers all information of the E-based Adams spectral sequence in ordinary spectra as well.

Having established the machinery of Adams filtrations, we will now describe how they help us understand νE-complete objects in synthetic spectra in terms of E-complete spectra. We first recall what it means for an object to be complete with respect to some Adams type homology theory (or more generally, any dualisable homotopy associative algebra).

Definition 8.7. A spectrum X is E-complete, or more correctly E-nilpotent complete, if the canonical map
$$X \to X_E^\wedge := \lim_{\Delta} X \wedge E^{\wedge \bullet}$$
to its E-completion induced by the augmentation obtained from the unit map is an equivalence. Note that the cosimplicial object of which we consider the limit is simply the cobar resolution of E smashed with X. Further, note that this extends easily to other homotopy theories such as synthetic spectra, whence there is a notion of νE-complete synthetic spectra, which we will see is the same as the notion of completeness established in the section on Dwyer–Greenlees theory. Indeed, we saw that the completion functor could be described precisely in terms of the cosimplicial object above.

Fortunately, now that we have developed Adams filtrations, there is a different way of verifying this. This relies on the ∞-categorical Dold–Kan correspondence

$$\begin{aligned} \mathcal{F}\mathrm{un}(\Delta, \mathcal{C}) &\xrightarrow{\sim} \mathcal{F}\mathrm{un}(\mathbb{Z}^{\mathrm{op}}_{\geq 0}, \mathcal{C}), \\ C^\bullet &\mapsto \lim_{\Delta^{\leq \star}} C^\bullet \end{aligned}$$

between cosimplicial objects and towers in a stable ∞-category $\mathcal{C}$. The functor depicted above sends a cosimplicial object to the tower obtained by taking its limits over the full subcategories of Δ on objects of cardinality no greater than some upper bound. Letting this upper bound vary gives us the tower structure on the latter. This is a well established result, appearing in [Lur17] Theorem 1.2.4.1.

To follow the standard convention, we denote the limit of a cosimplicial object over $\Delta^{\leq\star}$ by its partial totalisation $\mathrm{Tot}_\star$, with the full limit being referred to as its totalisation Tot, dually to the geometric realisation. We can therefore adopt the notation

$$C^\bullet \mapsto \mathrm{Tot}_\star C^\bullet$$

for the Dold–Kan correspondence.

Lemma 8.6. *If we view the cobar resolution $E^{\wedge\bullet}$ of E as a cosimplicial object in* Sp*, the Dold–Kan equivalence sends this to the tower* $\mathrm{cof}(\mathbb{S}_{\star+1} \to \mathbb{S})$*, where $\mathbb{S}_{\star+1}$ is the $(\star+1)$'st term in the E-based Adams filtration of $\mathbb{S}$.*

Proof. Using the formula above, we see that the proof of this statement essentially reduces to a computation of the partial totalisations of the cosimplicial object $E^{\wedge\bullet}$. In fact, for the purpose of computing limits indexed by $\Delta^{\leq n}$, we see that it suffices to consider the subcategory $\mathcal{P}([n])$ on nonempty subsets of $[n]$ with the induced ordering, and inclusions between them making this into a poset. The observation that this inclusion is right cofinal is made in [Lur17] Lemma 1.2.4.17. The cobar resolution then becomes a functor[11]

$$\begin{aligned} \mathrm{cb}(\mathbb{S} \to E) : \mathcal{P}([n]) &\to \mathrm{Sp}, \\ S \subset [n] &\mapsto \bigwedge_{i \in S} E \wedge \bigwedge_{j \notin S} \mathbb{S}. \end{aligned}$$

We now see that given some inclusion of subset $S \subset S'$, the induced map between values of $\mathrm{cb}(E)$ is the map

$$\bigwedge_{i\in S} E \wedge \bigwedge_{j \notin S} \mathbb{S} \simeq \bigwedge_{i \in S} E \wedge \bigwedge_{k \in S' \setminus S} \mathbb{S} \wedge \bigwedge_{j \notin S'} \mathbb{S} \to \bigwedge_{i \in S'} E \wedge \bigwedge_{j \notin S'} \mathbb{S}$$

that applies the unit map $\mathbb{S} \to E$ at every index $k \in S' \setminus S$. We make two observations about this.

- This extends to a functor $\mathrm{cb}_+(\mathbb{S} \to E)$ from $\mathcal{P}_+([n])$, the poset of possibly empty subsets and inclusions between them. Of course this is none other than the left cone

  $$\mathcal{P}_+([n]) = \mathcal{P}([n])^{\triangleleft}.$$

  In particular, we note that the value at $\varnothing$ of a functor in $f_+ \in \mathrm{Sp}^{\mathcal{P}_+([n])}$ is a cone over the restricted diagram in $f \in \mathrm{Sp}^{\mathcal{P}_+([n])}$, so that by the universal property of the limit there is always a map

  $$f_+(\varnothing) \to \lim_{\mathcal{P}([n])} f.$$

- Further, this operation can be done for $\mathbb{S} \to E$ replaced by any $[n]$-indexed collection of morphisms, as in [MNN17] Proposition 2.10. In particular, we can consider the same operation induced by the constant family on the map

  $$\bar{E} \to 0$$

  appearing in the fibre sequence

  $$\bar{E} \to \mathbb{S} \to E.$$

[11] The admittedly clunky notation $\bigwedge$ simply refers to iterated smash products and not some exterior algebra.

Since there is an obvious map

$$[\bar{E} \to 0] \to [\mathbb{S} \to E]$$

in Sp^{Δ^1} appearing in the Cartesian square defining $\bar{E}$, we use the two remarks above to obtain a natural transformation of functors

$$\mathrm{cb}_+(\bar{E} \to 0) \to \mathrm{cb}_+(\mathbb{S} \to E) \in \mathrm{Sp}^{\mathcal{P}_+([n])}.$$

Since cofibres are computed levelwise in this functor ∞-category, we see that the cofibre of this map is a functor which evaluates at $\varnothing \subset [n]$ to

$$\mathrm{cof}(\bigwedge_{i\in[n]} \bar{E} \to \bigwedge_{i\in[n]} \mathbb{S}).$$

By our observation above, this forms a cone over the restricted diagram, whence there is a map to the limit of the latter. Now the restriction can be described as evaluating at every nonempty $S \subset [n]$ to the cofibre

$$\mathrm{cof}(\bigwedge_{i\in S} \bar{E} \wedge \bigwedge_{j\notin S} 0 \to \bigwedge_{i'\in S} E \wedge \bigwedge_{j'\notin S} \mathbb{S}) \simeq \mathrm{cb}(\mathbb{S} \to E)(S),$$

since the source of this cofibre is trivial as soon as we include a smash factor of 0. Therefore, the restriction of this cofibre is none other than our original cobar resolution $\mathrm{cb}(\mathbb{S} \to E)$, and the observation above gives a map

$$\mathrm{cof}(\bigwedge_{i\in[n]} \bar{E} \to \bigwedge_{i\in[n]} \mathbb{S}) \to \lim_{\mathcal{P}([n])} \mathrm{cb}(\mathbb{S} \to E) \simeq \mathrm{Tot}_n E^{\wedge\bullet}.$$

To show that this is an equivalence, we note that the unit map $[\mathbb{S} \to E]$ actually sits in a cofibre sequence

$$[\bar{E} \to 0] \to [\mathbb{S} \to E] \to [E \to E],$$

with the first map being the one considered above. If we now replace $[\mathbb{S} \to E]$ by $[\bar{E} \to 0]$ in the limit comparison map above, it is clear that this becomes an equivalence since both sides are trivial. If we instead inserted the identity $[E \to E]$, we see that it is an equivalence by an application of [Lur17] Lemma 1.2.4.15. We then conclude that this comparison map is also an equivalence for the extension $[\mathbb{S} \to E]$.

Now we use our previous comment in the construction of the Adams tower, which said that it could equally well be obtained by successively smashing with the fibre $\bar{E}$. In particular, we obtain

$$\mathbb{S}_{\star+1} \simeq \mathbb{S} \wedge \bar{E}^{\wedge\star+1} \simeq \bigwedge_{i\in[\star]} \bar{E},$$

whence the explicit description of $\mathrm{Tot}_\star E^{\wedge\bullet}$ in terms of the cofibre above precisely recovers what needed to be shown. □

In fact, the construction of the Adams tower and the description of the tower obtained from the cobar resolution of E through the Dold–Kan equivalence can be stated in great generality, similar to that of Section 4.5. Continuing within the vein of that section, we use the Adams tower to obtain another description of completeness.

Proposition 8.4. *Let $\mathcal{C}$ be a stable presentably symmetric monoidal ∞-category with a dualisable commutative algebra object inside it, in this case $\mathcal{C}$ is* Sp *or* $\mathcal{S}\mathrm{yn}_E$ *and A is E resp. νE, and we adopt the notation of the former. Then the following are equivalent for an object X:*

- *X is E-complete in the sense that it lies in the right orthogonal complement of the A-trivial objects.*
- *X is such that the augmentation*

$$X \to \mathrm{Tot}(X \wedge E^{\wedge\bullet})$$

  is an equivalence.
- *X is such that the limit of the E-Adams tower of X vanishes.*

Proof. Note that the first two items are equivalent by our previous result Lemma 4.2. We therefore only need to show that the last two are equivalent. In one direction, assume that X is equivalent to the limit of $X \wedge E^{\wedge \bullet}$. The latter cosimplicial object is equivalent to the tower $X \wedge \mathrm{cof}(\bar{E}^{\wedge \star+1} \to \mathbb{S})$ by the Dold–Kan equivalence in Lemma 8.6. By presentability, we see that this tower can also be described by

$$\mathrm{cof}(X \wedge \bar{E}^{\wedge \star+1} \to X).$$

This fits inside a cofibre diagram of towers

$$\begin{tikzcd}
\cdots \arrow[r] & X \wedge \bar{E}^{n+1} \arrow[r] \arrow[d] & X \wedge \bar{E}^{n} \arrow[r] \arrow[d] & \cdots \arrow[r] & X \wedge \bar{E} \arrow[d] \\
\cdots \arrow[r] & X \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & \cdots \arrow[r] & X \arrow[d] \\
\cdots \arrow[r] & \mathrm{cof}(X \wedge \bar{E}^{\wedge n+1} \to X) \arrow[r] & \mathrm{cof}(X \wedge \bar{E}^{\wedge n} \to X) \arrow[r] & \cdots \arrow[r] & \mathrm{cof}(X \wedge \bar{E} \to X).
\end{tikzcd}$$

Now note that the top tower is precisely the E-Adams tower of X (up to a shift). Indeed, we saw that this tower was constructed iteratively by smashing on fibres of the unit map $\mathbb{S} \to E$. By stability, we see that cofibres commute with limits, so that this cofibre sequence of towers induces a cofibre sequence on the limits of the form

$$\lim(X \wedge E^{\wedge \star}) \to X \to \lim(\mathrm{cof}(X \wedge E^{\wedge \star+1} \to X)) \simeq \lim(X \wedge E^{\wedge \bullet}),$$

where the last line uses the Dold–Kan equivalence. We conclude that the last map is an equivalence if and only if the fibre is zero, and by our reasoning above this is the case if and only if the E-based Adams tower of X has a vanishing limit. □

The upshot of the description above, is that we have previously established a relation between E-Adams towers in spectra and νE-Adams towers in synthetic spectra, twisted by τ. We can therefore hope to relate E-completeness in spectra with νE-completeness in synthetic spectra. In fact, this is well known for synthetic analogues, forming the content of [BHS19] Proposition A.13, which we review here.

Proposition 8.5. *Let X be a spectrum, then the following are equivalent.*

- *X is E-complete.*
- *νX is νE-complete.*
- *νX is τ-complete.*

Proof. The first two can be compared easily, since the previous proposition can be applied, whence it suffices to show that the E-Adams tower of X has vanishing limit if and only if the νE-Adams tower of νX has vanishing limit. Recall that the νE-Adams tower of νX is of the form

$$(\nu X)_n = \Sigma^{0,n} \nu X_n.$$

This means that we would like to compare the limits

$$\lim_n \Sigma^{0,n} \nu X_n, \qquad \lim_n X_n.$$

For this, let us rephrase these in terms of a right adjoint functor, namely the spectral Yoneda embedding Y. Since Y preserves limits, we have an equivalence

$$Y(\lim_n X_n) \simeq \lim_n Y(X_n),$$

so that we can embed the E-based Adams tower of X into the synthetic realm. As for the νE-based Adams tower of νX, we use the equivalence

$$\Sigma^{0,n} \nu X_n \simeq \tau_{\geq -n} Y(X_n).$$

Indeed, we saw that there was a cofibre sequence

$$\Sigma^{0,-1}\nu X \to \nu X \to C\tau \otimes \nu X$$

decomposing a synthetic analogue into its 1-connective, resp. 0-coconnective parts. Using Lemma 8.7 to identify νX with the connective cover of $Y(X)$, we can desuspend in the deformation direction n times to identify

$$\Sigma^{0,n}\nu X_n \simeq \tau_{\geq -n} Y(X_n).$$

In particular, the inclusion of $(-n)$-connective covers induces a cofibre sequence

$$\tau_{\geq -n} Y(X_n) \to Y(X_n) \to \tau_{\leq -n}(X_n),$$

which induces a map of towers as n varies. In particular, by stability this will commute with the limit, so that we obtain a cofibre sequence on the limits of each tower of the form

$$\lim_n \tau_{\geq -n} Y(X_n) \simeq \lim_n \Sigma^{0,n}\nu X_n \to \lim_n Y(X_n) = Y(\lim_n X_n) \to 0,$$

where the last terms vanishes since the t-structure on synthetic spectra is right complete, i.e. there are no ∞-coconnective objects. In particular, this means that the limits above are equivalent. Using the fact that Y is a fully faithful embedding, we see that νX is νE-complete if and only if X is E-complete, since one limit vanishes if and only if the other does.

Finally, let us check that νE-completeness is equivalent to τ-completeness. Once again, recall that νE-completeness is equivalent to the limit of the νE-Adams tower of νX vanishing. Similarly, since τ-completeness is equivalent to being τ-torsion by our discussion of Dwyer–Greenlees theory, we can test τ-completeness on the vanishing of the limit of the tower obtained by repeated multiplication by τ. Therefore, we can fit νX into a double tower

$$\begin{tikzcd}
 & \vdots \arrow[d, "\tau"] & \vdots \arrow[d, "\tau"] & \vdots \arrow[d, "\tau"] \\
\cdots \arrow[r] & \nu X_2 \arrow[r] \arrow[d, "\tau"] & \Sigma^{0,-1}\nu X_1 \arrow[r] \arrow[d, "\tau"] & \Sigma^{0,-2}\nu X \arrow[d, "\tau"] \\
\cdots \arrow[r] & \Sigma^{0,1}\nu X_2 \arrow[r] \arrow[d, "\tau"] & \nu X_1 \arrow[r] \arrow[d, "\tau"] & \Sigma^{0,-1}\nu X \arrow[d, "\tau"] \\
\cdots \arrow[r] & \Sigma^{0,2}\nu X_2 \arrow[r] & \Sigma^{0,1}\nu X_1 \arrow[r] & \nu X.
\end{tikzcd}$$

In this diagram, the bottom horizontal row is the νE-Adams resolution of νX, and higher horizontal rows are obtained as its iterated $\Sigma^{0,-1}$-suspensions, with arrows between horizontal layers given by the action of τ. We see that the limit of the rightmost vertical tower vanishes if and only if νX is τ-complete, while the limit of the bottom horizontal tower vanishes if and only if νX is νE-complete. Now the main result is to show that both of these limits are precisely equivalent to the limit of this entire diagram. Concretely, we claim that there is an equivalence

$$\lim_n \Sigma^{0,n}\nu X_n \simeq \lim_n \lim_m \Sigma^{0,n-m}\nu X_n \simeq \lim_m \lim_n \Sigma^{0,n-m}\nu X_n \simeq \lim_m \Sigma^{0,-m}\nu X.$$

The middle equivalence is benign, as it consists merely of a limit exchange. Let us tackle the second equivalence.

This actually underlies a slightly stronger result, namely that after taking the limit over m, we obtain an essentially constant tower, whose limit (ergo also its value at every level) is then $\lim_m \Sigma^{0,-m}\nu X$. Indeed, note that per construction we have a precise description of the cofibre at every level of the Adams tower: it is given by $\Sigma^{0,n-m}\nu X_n \otimes \nu E$ at height m. Taking the limit over m, we see that the successive cofibres are given by

$$\lim_m \Sigma^{0,n-m}(\nu X_n \otimes \nu E) \simeq \lim_m \Sigma^{0,-m}(\Sigma^{0,n}\nu X_n \otimes \nu E).$$

If we can show that these vanish, we will have shown that the Adams tower at height $m \to \infty$ is essentially constant. In particular, given the explicit description above in terms of a limit over repeated application of τ (followed by a conservative functor), this is equivalent to showing that every

$$\Sigma^{0,n}\nu X_n \otimes \nu E = \Sigma^{-n}(\nu\mathbb{S}^n \otimes \nu X_n \otimes \nu E) \simeq \Sigma^{-n}(\nu(\Sigma^n X_n) \otimes \nu E)$$

is τ-complete. Since this subcategory is closed under (co)limits, we can ignore the suspension on the outside and it suffices to show that for a general spectrum Y, the object $\nu Y \otimes \nu E$ is τ-complete. Recall from Proposition 8.3 that synthetic analogues of homotopy E-modules such as $Y \wedge E$ were nice connective objects in the sense that their bigraded homotopy groups vanish in negative Chow degree. Recalling that τ controls the t-structure on $\mathcal{S}\mathrm{yn}_E$, it is perhaps not surprising to see that these are τ-complete for the purely formal reason sketched here.

Once again, it suffices to show that the limit of τ-multiplication tower of this object vanishes, and this can now be shown using the property of synthetic analogues of homotopy E-modules sketched above. Indeed, recall that $\mathcal{S}\mathrm{yn}_E$ is generated under colimits by synthetic analogues of finite E-projectives, so that it suffices to fix such a P and show that the mapping spectrum

$$\mathrm{Map}(\nu P, \lim_m \Sigma^{0,-m}\nu Y \otimes \nu E)$$

vanishes. Now this mapping spectrum can be rewritten as

$$\begin{aligned}
\mathrm{Map}(\nu P, \lim_m \Sigma^{0,-m}\nu Y \otimes \nu E) &\simeq \lim_m \mathrm{Map}(\nu P, \Sigma^{0,-m}\nu Y \otimes \nu E), \\
&\simeq \lim_m \mathrm{Map}(\mathbb{S}^{0,m}, \nu(Y \wedge E \wedge DP)), \\
&\simeq 0.
\end{aligned}$$

To go the second line, we used the dualisability of synthetic analogues of finite E-projective spectra–a consequence of the monoidality of the synthetic analogue when restricted to these spectra. We then used the observation that the mapping spectrum appearing inside the limit computes the synthetic homotopy groups of the synthetic analogue inside it. Now by Proposition 8.3 concerning the connectivity of bigraded homotopy groups of the synthetic analogue of a homotopy E-module, this resulting spectrum is m-connective. Since there are no nontrivial ∞-connective spectra, we see that taking the limit as m increases gives rise to the desired vanishing.[12]

We conclude that $\nu Y \otimes \nu E$ is τ-complete for any spectrum Y, so that the successive cofibres of the Adams tower at height $m \to \infty$ are trivial, whence it is constant. Its limit is then equal to its 0-th term, which is simply the limit

$$\lim_m \Sigma^{0,-m}\nu X,$$

and we conclude.

Now we proceed to show the first equivalence by a similar reasoning. Indeed, the successive cofibres in the tower of repeated τ-multiplication are also known: in height m, it is given simply by the cofibre of τ acting on the m-th object, i.e. in height m and Adams degree n it is $C\tau \otimes \Sigma^{0,n-m}\nu X_n$. Once again, we will show that these vanish, but this time in the limit in the Adams direction, i.e. for $n \to \infty$. We have therefore reduced our problem to showing that for every spectrum Y, the synthetic spectrum $C\tau \otimes \nu Y$ is νE-complete, since this recovers the vanishing of the cofibres in the limit above.

As is routine by now, this will be done by showing that the limit of the νE-Adams tower of $C\tau \otimes \nu Y$ vanishes. We remarked earlier that the formation of the Adams tower is monoidal in the sense that the Adams tower for $C\tau \otimes \nu Y$ can be obtained by tensoring the Adams tower of νY with $C\tau$ levelwise. Concretely, this means that the n-th part of the Adams tower of $C\tau \otimes \nu Y$ is given by

$$(C\tau \otimes \nu Y)_n \simeq C\tau \otimes (\nu Y)_n = C\tau \otimes \Sigma^{0,n}\nu Y_n,$$

[12] Compare the proof of Proposition 9.2 for a similar consideration of the connectivity of these mapping spectra computing bigraded synthetic homotopy groups.

where Y_n is the n-th term in the E-based Adams tower of the spectrum Y. But now we see that the latter can be written out as

$$C\tau \otimes \Sigma^{0,n}\nu Y_n \simeq \Sigma^{-n}(C\tau \otimes \nu(\Sigma^n Y_n)).$$

Now use the observation that tensoring with $C\tau$ takes the 0-coconnective part, from Corollary 8.1, to see that the resulting object is $(-n)$-coconnective. If we now take the limit $n \to \infty$, computing the limit of the νE-based Adams tower of $C\tau \otimes \nu X$, we see that the limit must be a $(-\infty)$-coconnective synthetic spectrum, hence trivial by right completeness of the t-structure on synthetic spectra. We conclude that in the limit $n \to \infty$, the tower obtained by repeatedly applying τ is essentially constant, so that its limit over m is equivalent to its zeroeth part, which is none other than the limit of the νE-based Adams tower of X

$$\lim_n \lim_m \Sigma^{0,n-m}\nu X_n \simeq \lim_n \Sigma^{0,n}\nu X_n.$$

Bringing it all together, we use the chain of equivalences that was shown in the previous paragraphs to see that the limit of the νE-based Adams tower of X vanishes if and only if the limit of its multiplication-by-τ tower vanishes. By the results of our analysis of Adams towers and Dwyer–Greenlees theory, we see that these (equivalent) conditions are themselves equivalent to νX being νE-complete, resp. τ-complete. □

8.4 Realisation and τ-inversion

An important part of the theory of synthetic spectra, is that one can explicitly identify the subcategories

$$\mathcal{S}\mathrm{yn}_E[\tau^{-1}] \simeq \mathrm{Sp}, \qquad \mathrm{Mod}(\mathcal{S}\mathrm{yn}_E; C\tau) \simeq \mathcal{S}\mathrm{table}_{E_*E}.$$

As is apparent from the section on deformations, and as will be elaborated on below; these computations are essential to the theory of deformations, indeed they are the identifications of the generic and special fibres.

The identification of the generic fibre uses the spectral Yoneda embedding

$$Y : \mathrm{Sp} \to \mathcal{S}\mathrm{yn}_E.$$

First, let us note that it takes values in the full subcategory of τ-invertible synthetic spectra. In fact, we can explicitly identify which synthetic spectrum some spectral Yoneda embedding $Y(X)$ is the τ-inversion of.

Lemma 8.7 ([Pst18] Proposition 4.36). *The canonical map*

$$\nu X \to Y(X)$$

is a connective cover and a τ-inversion.

Note that by the canonical map we mean the map of sheaves induced by the levelwise connective cover

$$\nu X(P) \simeq \Sigma^\infty \mathrm{map}(P, X) \simeq \mathrm{Map}(P, X)_{\geq 0} \to \mathrm{Map}(P, X) \simeq Y(X)(P).$$

Proof. By the explicit description of the comparison map above as arising from a levelwise connective cover, we see that its fibre is (-1)-coconnective at every finite E-projective P. Since the t-structure on synthetic spectra is such that the coconnective part consists precisely of sheaves of coconnective spectra, we deduce that the fibre of the canonical map above is (-1)-coconnective as a synthetic spectrum. Combining this with the observation that νX is connective as a synthetic spectrum, we obtain the result.

To further identify this comparison map with a τ-inversion (i.e. the unit transformation associated to the reflective subcategory of τ-invertible spectra) it suffices to note that the fibre of this canonical morphism vanishes after τ-inversion. For this, we note more generally that any n-coconnective synthetic spectrum X has trivial τ-inversion for $n \in \mathbb{Z}$. Indeed, the latter is computed as

$$\tau^{-1}X \simeq \mathrm{colim}\,(X \xrightarrow{\tau} \Sigma^{0,1}X \xrightarrow{\tau} \Sigma^{0,2}X \xrightarrow{\tau} \cdots,)$$

but we see that every $\Sigma^{0,n}X$ is $(k-n)$-coconnective since $\mathbb{S}^{0,n} = \Sigma^{0,-n}\nu\mathbb{S}^n$ so that this suspension smashes with a connective object and shifts the connectivity down by n degrees. Therefore, as n tends to infinity, the colimit must be infinitely coconnective, hence trivial since the t-structure on synthetic spectra is right complete. □

This means that there is a factorisation of Y as

$$Y : \mathrm{Sp} \to \mathcal{S}\mathrm{yn}_E[\tau^{-1}] \to \mathcal{S}\mathrm{yn}_E.$$

We now claim that this is an equivalence.

Proposition 8.6 ([Pst18] Theorem 4.37). *The induced functor*

$$Y : \mathrm{Sp} \to \mathcal{S}\mathrm{yn}_E[\tau^{-1}]$$

is an equivalence

Proof. We begin by showing that Y is essentially surjective. For this, let us first note that Y has a right adjoint, which we will denote Y^R. Indeed, since Y is constructed by sending a spectrum to the presheaf defined by taking mapping spectra out of finite (E-projective) ergo compact spectra, it commutes with filtered colimits. To show that it is cocontinuous it suffices to note that it commutes with limits, hence by stability commutes with finite colimits as well. Since τ-invertible spectra form a reflective and coreflective subcategory of $\mathcal{S}\mathrm{yn}_E$ (e.g. by noting that it fulfils the conditions of smashing localisation in the sense of Definition 4.9), we see that the factorisation of Y through τ-invertible synthetic spectra is (co)continuous as well. By presentability of both Sp and $\mathcal{S}\mathrm{yn}_E[\tau^{-1}]$, it must have the aforementioned right adjoint Y^R. Now we claim that given some τ-invertible synthetic spectrum X, we can recover X as the spectral Yoneda embedding of $Y^R X$, or more specifically that the counit transformation

$$YY^R X \to X$$

is an equivalence. By general considerations of adjunctions, we see that it suffices to show that the right adjoint is conservative for this counit to be a natural equivalence. However, since both source and target are stable, it suffices to show that fibres of morphisms are trivial if and only if their image under Y^R is, i.e. that if F is a synthetic spectrum, $F \simeq 0$ if and only if $Y^R \simeq 0$. To show the only nontrivial implication, we note that by the adjunction $Y \dashv Y^R$, the spectrum $Y^R F$ being zero implies that for any finite E-projective P we have

$$\begin{aligned} 0 &\simeq \mathrm{map}(P, Y^R F), \\ &\simeq \mathrm{map}(Y(P), F), \\ &\simeq \mathrm{map}(\nu P, F)), \\ &\simeq \Omega^\infty F(P). \end{aligned}$$

The third equivalence uses the previous observation that $Y(P)$ is the τ-inversion of νP, allowing us to apply the adjunction between τ-inversion and the inclusion of τ-invertible synthetic spectra. From the equivalences above, we see that the spectrum $F(P)$ is at least (-1)-coconnective, so that F is an n-coconnective synthetic spectrum for $n \leq -1$. By the observation made in the proof of Lemma 8.7, we see that it is trivial as a τ-invertible synthetic spectrum. We conclude that

$$Y : \mathrm{Sp} \to \mathcal{S}\mathrm{yn}_E[\tau^{-1}]$$

is essentially surjective.

To show that it is fully faithful, we apply a similar reasoning using the universal property of the synthetic analogue and the τ-inversion functor to note that for any spectrum T and integer n there is an equivalence

$$\begin{aligned} \mathrm{map}(\mathbb{S}^n, T) &\simeq \Omega^\infty \mathrm{Map}(\mathbb{S}^n, T), \\ &\simeq \Omega^\infty Y(T)(\mathbb{S}^n), \\ &\simeq \mathrm{map}(\nu\mathbb{S}^n, Y(T)), \\ &\simeq \mathrm{map}(Y(\mathbb{S}^n), Y(T)). \end{aligned}$$

This means that Y is fully faithful when the source of a mapping space is a sphere. However, we noted above that Y so cocontinuous, so that any spectrum can be expressed as a colimit of spheres, leading to a colimit in the first input of the mapping space above, which comes out as a limit, so that Y is fully faithful on all of Sp. More accurately, the subcategory of spectra C such that $\mathrm{map}(C, -) \simeq \mathrm{map}(Y(C), Y(-))$ is closed under colimits and contains all spheres, hence is all of Sp. □

Remark 8.10. If one restricts to finite E-projective spectra, we see that there is a factorisation

$$\begin{tikzcd} \mathrm{Sp}_E^\omega \arrow[r, hook] \arrow[dr, "\nu"'] & \mathrm{Sp} \arrow[r, "Y", "\sim"'] & \mathcal{S}\mathrm{yn}_E[\tau^{-1}]. \\ & \mathcal{S}\mathrm{yn}_E \arrow[ur, "\tau^{-1}"'] & \end{tikzcd}$$

so that the functor τ^{-1} is a left Kan extension of the (symmetric monoidal) inclusion of finite E-projective spectra along the Yoneda embedding (in the form of the restricted synthetic analogue), hence is symmetric monoidal by the universal property of the Day convolution. Since $\mathcal{S}\mathrm{yn}_E[\tau^{-1}]$ is a symmetric monoidal localisation since as is smashing, we see that τ^{-1} restricted to this full subcategory is a symmetric monoidal inverse to Y, whence Y is symmetric monoidal as well. We conclude that the equivalence

$$Y : \mathrm{Sp} \xrightarrow{\sim} \mathcal{S}\mathrm{yn}_E[\tau^{-1}]$$

is symmetric monoidal.

9 Synthetic spectra as a deformation

In this final section, we bring together the theory of filtered spectra and deformations as developed along the course of this work to give a description of synthetic spectra as a deformation. This was first observed in [BHS19], and we obtain the same result as in op. cit. while working out some terse proofs and fitting it within our framework of deformations. There are two ways of equipping synthetic spectra with the structure of a deformation, corresponding to our two notions of deformations. On one hand, one could fit them into the framework of Section 7.3, which would amount to specifying a collection of graded objects and a pair of functors to and from spectra. This will prove rather easy, as the construction of synthetic spectra as well as this definition of deformations are set up so that this is immediate. On the other hand (after completing at everything at a prime p), when E is MU, we can go the other way around, using that even MU-based synthetic spectra are equivalent to cellular complex motivic spectra by [Pst18], and using the description of the latter in terms of modules over a certain algebra object in filtered spectra due to [Ghe+18]. It is then shown in [Gre21] that these modules in filtered spectra arise as quasicoherent sheaves on a nonconnective spectral stack that admits the structure of a one-parameter deformation. One then proceeds to describe the generic and special fibres of this deformation, and notes that they agree with spectra and the algebraic fibre respectively. The downside of the latter approach is that it is only described for $E = \mathrm{MU}$ and after p-completion, passing through two previously established results relating MU-based synthetic spectra (up to an additional evenness condition) and motivic spectra. Since we are not in a position to recall nonconnective spectral algebraic geometry or complex motivic homotopy theory, we will not elaborate on this deformation picture, and rather focus on the result of Burklund–Hahn–Senger equipping $\mathcal{S}\mathrm{yn}_E$ with the structure of a deformation. To fully fit this in the framework above, we require that this deformation be monadic, i.e. that $\mathcal{S}\mathrm{yn}_E$ is generated by its bigraded spheres. In general, the cellular objects–those generated by the bigraded spheres–only form a full subcategory denoted $\mathcal{S}\mathrm{yn}_E^{\mathrm{cell}}$. Therefore, to describe synthetic spectra as a deformation one ought to restrict to cellular objects. The following subsection illustrates when this restriction is trivial.

9.1 Cellularity

As illustrated above, we require our ∞-category of synthetic spectra to be generated by the bigraded spheres $\mathbb{S}^{t,w}$. For this reason, we actually restrict to the full subcategory on cellular objects generated by these. Fortunately, in many cases of interest such as $E = \mathbb{F}_p$ or $E = \mathrm{MU}$ (cf. [Pst18] Section 6.1), the cellular subcategory is the entire ∞-category of synthetic spectra. Henceforth, when E is MU or $\mathbb{F}_p$ will no longer mention this cellular restriction and leave it implicit, as it will be trivial.

To prove this fact, we will primarily follow the short argument given in [Pst18], filling in the references to more classical work about the algebra of MU_* as in [CS69]. The fundamental observation is that MU_* is a connected $\mathbb{Z}$-algebra, i.e. a positively graded algebra over $\mathbb{Z}$ with an augmentation $\mathrm{MU}_* \to \mathbb{Z}$ inducing an equivalence in degree zero. This is immediate from the classical description of the graded ring MU_* as

$$\mathrm{MU}_* \cong \mathbb{Z}[x_i \mid i \geq 1, |x_i| = 2i],$$

so that the augmentation is given by modding out the x_i's and the connectedness is obvious since all of these live in strictly positive degrees. This has some pleasant algebraic properties.

Lemma 9.1. *Since MU_* is a connected $\mathbb{Z}$-algebra, the functor $\mathbb{Z} \otimes_{\mathrm{MU}_*} -$ given by base change along the augmentation map is such that it reflects epimorphisms.*

Proof. It is sufficient to show that the functor above detects zero objects. Indeed, one could then apply this result to the cokernel of a morphism of MU_*-modules f to conclude that f is an epimorphism (i.e. has vanishing cokernel) if and only if $\mathbb{Z} \otimes_{\mathrm{MU}_*} f$ is an epimorphism, one direction being induced by the fact that tensor products preserve cokernels. Now the proof of the aforementioned statement is rather immediate from the definition of connectedness. Indeed, suppose that M is an MU_*-module such that

$$\mathbb{Z} \otimes_{\mathrm{MU}_*} M \cong 0.$$

Letting I denote the augmentation ideal of MU_*, i.e. the kernel of the augmentation, one can rephrase the property above as stating that

$$0 \cong \mathbb{Z} \otimes_{\mathrm{MU}_*} M \cong \operatorname{coker}(I \to \mathrm{MU}_*) \otimes_{\mathrm{MU}_*} M,$$

so that this is equivalent to the map

$$I \otimes M \to M$$

being an epimorphism. Since MU_* is connected, the augmentation is an equivalence in degree zero, whence $I_0 = 0$ and $(I \otimes M)_0 = 0$ and we conclude that $M_0 = 0$ as well. Now we can induct up, by noting that if M is concentrated in degrees $\leq k$ for some $k \geq 0$, then $I \otimes M$ is concentrated in degrees $\leq k+1$, one being the lowest degree in which one can find nonzero elements of I[13]. We conclude that M is surjected upon by a zero module in every degree $\leq k+1$, hence is zero by induction. □

An immediate consequence of this is that projective MU_*-modules are quite simple.

Corollary 9.1. *Any projective MU_*-module is free, so the two conditions are equivalent.*

Proof. Let M be a projective MU_*-module, then it can be written as a direct summand of a free module

$$M \oplus N \cong F.$$

Now the base change along the augmentation respects this coproduct, and since MU_* gets sent to $\mathbb{Z}$, we see that $\mathbb{Z} \otimes_{\mathrm{MU}_*} M$ is a direct summand of a free $\mathbb{Z}$-module, hence projective itself ergo free. If we let $\{b_\alpha \mid \alpha \in A\}$ be a basis for this free $\mathbb{Z}$-module, we obtain a basis for the free MU_*-module $\mathrm{MU}_* \otimes_{\mathbb{Z}} \mathbb{Z} \otimes_{\mathrm{MU}_*} M$. Now define an MU_*-linear map from the latter back down to M by sending some basis element $1 \otimes b_\alpha$ to the element c_α in M such that $1 \otimes c_\alpha$ represents the class of b_α in $\mathbb{Z} \otimes_{\mathrm{MU}_*} M$. It is clear that this is well defined, and an epimorphism after base change along the augmentation. Using the lemma established above, we conclude that it is an epimorphism. By projectivity of M it has a section, which becomes an equivalence after base change, whence by right cancellation we conclude that this map is an isomorphism and M is free. □

When these MU_*-modules arise from finite spectra, we can interpret the structure above as being a statement about MU-homology and integral homology. This is captured in the following proposition.

Proposition 9.1. *For X a finite spectrum, the following are equivalent*:

1. *The module MU_*X is projective over MU_*.*
2. *The module MU_*X is free over MU_*.*
3. *The integral homology $\mathbb{Z}_*X = \mathrm{H}(X;\mathbb{Z})$ of X is free over $\mathbb{Z}$.*

Further, the rank of MU_*X over MU_* is equal to the rank of $\mathrm{H}_*(X;\mathbb{Z})$ over $\mathbb{Z}$.

Proof. The equivalence between the first two points is an immediate consequence of our previously established corollary. The equivalence between the second and third items follow from a more careful analysis of the Atiyah–Hirzebruch spectral sequence for complex cobordism. Note that this is given by

$$E^2 = \mathrm{H}_*(X;\mathrm{MU}_*) \implies \mathrm{MU}_*X.$$

Now MU_* is torsion-free as a $\mathbb{Z}$-module from its explicit description, so that the Künneth spectral sequence computing the E^2-term collapses, and it is given by the graded tensor product

$$E^2 = \mathrm{H}_*(X;\mathbb{Z}) \otimes \mathrm{MU}_*.$$

By our assumption on the integral homology of X, we see that this is a finite free MU_*-module, so that the lack of torsion causes the Atiyah–Hirzebruch spectral sequence to collapse at the E^2-page, whence we

[13] The proof we give here holds for any connected algebra over any ground ring, so that the degree one shows up, but in our case it is clear that I is generated in even degrees so that this can be strengthened to degree two.

conclude that MU_*X is simply $\mathrm{H}_*(X;\mathbb{Z}) \otimes \mathrm{MU}_*$, hence is free and of the same rank as the integral homology of X.

In the other direction, assuming that MU_*X is free, we want to show that $\mathrm{H}_*(X;\mathbb{Z})$ does not contain any torsion. If a torsion class existed in the latter, it would survive to an infinite cycle, hence lie in the image of the edge map

$$\mathbb{Z} \otimes_{\mathrm{MU}_*} \mathrm{MU}_*X \to \mathrm{H}_*(X;\mathbb{Z}),$$

where we have tensored the target up to $\mathbb{Z}$ by the left adjointness of the former to the forgetful functor. We will not elaborate on the details of this classical result, and refer to the construction and analysis of the bordism spectral sequence in [CS69] for a description of this process. At this point, recall that MU_* is a connected $\mathbb{Z}$-algebra, so that tensoring over the augmentation induces a rational equivalence

$$\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z} \otimes_{\mathrm{MU}_*} \mathrm{MU}_*X \cong \mathrm{H}(X;\mathbb{Q}).$$

Now MU_*X was assumed to be a free MU_*-module, so that $\mathbb{Z} \otimes_{\mathrm{MU}_*} \mathrm{MU}_*X$ is a free $\mathbb{Z}$-module and embeds into its rationalisation. Since the hypothetical torsion class is obviously trivial in $\mathrm{H}(X;\mathbb{Q})$, we pull back along the two injections to see that it was trivial in $\mathbb{Z} \otimes_{\mathrm{MU}_*} \mathrm{MU}_*X$ before applying the edge map, whence it is zero. We conclude that $\mathrm{H}^*(X;\mathbb{Z})$ has no torsion, hence is free. □

Equipped with this equivalence, we can return to the world of synthetic spectra to show that MU-synthetic spectra are cellular.

Lemma 9.2. *Consider the Adams type homology theory given by* MU. *Then the inclusion*

$$\mathcal{S}\mathrm{yn}_{\mathrm{MU}}^{\mathrm{cell}} \to \mathcal{S}\mathrm{yn}_{\mathrm{MU}}$$

is an equivalence, where the source is the full subcategory on cellular objects.

Proof. Recall that $\mathcal{S}\mathrm{yn}_{\mathrm{MU}}$ is generated by objects of the form νP for $P \in \mathrm{Sp}_{\mathrm{MU}}^{\omega}$. Therefore, it is sufficient to show that these are cellular in this case. The proof can be constructed by induction on the rank k of $\mathrm{H}_*(P;\mathbb{Z})$ over $\mathbb{Z}$. Indeed, since the former is free, we find a finite projective MU-module of rank $k-1$ by letting Q be the cofibre of the map

$$\mathbb{S}^{\ell} \to P$$

corresponding to the inclusion of a single free summand into $\mathrm{H}_{\ell}(P;\mathbb{Z})$ under the Hurewicz isomorphism, where ℓ is the lowest index in which the finite projective MU-module P has nontrivial singular homology.

It then suffices to show that this cofibre sequence lifts to synthetic analogues. This will be the case if the induced map

$$\mathrm{MU}_*P \to \mathrm{MU}_*Q$$

is a surjection by the characterisation of the Grothendieck topology on $\mathrm{Sp}_{\mathrm{MU}}^{\omega}$ established in the definition of synthetic spectra. This is immediately true since both terms are free MU_*-modules obtained by tensoring the integral homology with MU_*, and P' was obtained by quotienting out a free summand. We conclude that νP is an extension of νQ of rank $k-1$ and a synthetic sphere $\nu\mathbb{S}^{\ell} = \mathbb{S}^{\ell,\ell}$. □

Apart from MU, there is great computational interest in synthetic spectra based on the Eilenberg–Maclane spectra $\mathbb{F}_p$. In fact, $\mathbb{F}_p$-synthetic spectra are cellular as well by a slight generalisation of the proof above where we once again split off spheres using the Hurewicz isomorphism.

Lemma 9.3. *Consider the Adams type homology theory given by $\mathbb{F}_p$; Then the inclusion*

$$\mathcal{S}\mathrm{yn}_{\mathbb{F}_p}^{\mathrm{cell}} \to \mathcal{S}\mathrm{yn}_{\mathbb{F}_p}$$

is an equivalence.

Proof. In analogy with the proof of Lemma 9.2, we only need to show that the generators νP with $P \in \mathrm{Sp}^{\omega}_{\mathbb{F}_p}$ are cellular. Since such a spectrum P is finite, we can fix some index ℓ in which its lowest nontrivial homology group lives. In that case, the Hurewicz isomorphism once again gives us an isomorphism

$$\mathrm{H}_\ell(P;\mathbb{Z}) \cong \pi_\ell P.$$

Now we know that $\mathrm{H}_*(P;\mathbb{F}_p)$ is finite projective ergo free over $\mathbb{F}_p$, giving us an isomorphism

$$\mathrm{H}_\ell(P;\mathbb{F}_p) \cong \mathrm{H}_\ell(P;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F}_p.$$

This means that the integral homology of P must also be a free $\mathbb{Z}$-module, say of rank r equal to the rank of its $\mathbb{F}_p$-homology. We can then apply the same procedure, where we view the inclusion of a summand into $\mathrm{H}_\ell(P;\mathbb{Z})$ as representing a map of spectra

$$\mathbb{S}^\ell \to P \to Q$$

with cofibre $Q \in \mathrm{Sp}^{\omega}_{\mathbb{F}_p}$, whose rank over $\mathbb{Z}$ ergo $\mathbb{F}_p$ is $r-1$. Further,

$$H_*(P;\mathbb{F}_p) \to H_*(Q;\mathbb{F}_p)$$

is a surjection by construction. This means that the associated sequence of synthetic analogues

$$\nu\mathbb{S}^\ell \to \nu P \to \nu Q$$

is a cofibre sequence. We conclude that all generators of $\mathcal{S}\mathrm{yn}_{\mathbb{F}_p}$ are cellular. □

9.2 The deformation

We will now express $\mathcal{S}\mathrm{yn}_E$ as a deformation of homotopy theories, under the assumption that we have either restricted to cellular objects in $\mathcal{S}\mathrm{yn}_E$, or that $E = \mathbb{F}_p$ or MU, whence this restriction would be trivial. We have previously shown that geometric deformations and BHS deformations are both equivalent, so we can choose one model to apply. In fact, expressing $\mathcal{S}\mathrm{yn}_E$ in the latter model already uses a pretty fundamental result about this deformation, namely the identification of the generic fibre, in its construction, so we opt for the geometric route.

Theorem 9.1. *The ∞-category of synthetic spectra admits the structure of a deformation, i.e. a noncommutative stack over $\mathbb{A}^1/\mathbb{G}_m$, which is further $\mathbb{Z}$-plurigenic.*

Proof. This means that we want to endow $\mathcal{S}\mathrm{yn}_E$ with the structure of an $\mathrm{Sp}^{\mathrm{Fil}}$-algebra, which by the observations above can be encoded in a symmetric monoidal left adjoint

$$\mathrm{Sp}^{\mathrm{Fil}} \to \mathcal{S}\mathrm{yn}_E.$$

Once again, as in the comparison of the two models of deformations, we see that this arises simply from a symmetric monoidal functor

$$\mathbb{Z} \to \mathcal{S}\mathrm{yn}_E.$$

We let this functor be the one picking out the synthetic sphere $\mathbb{S}^{0,n}$ associated to an integer n. This is functorial by virtue of the comparison maps

$$\mathbb{S}^{0,n} \xrightarrow{\tau} \mathbb{S}^{0,n+1},$$

and is symmetric monoidal by virtue of the identity

$$\mathbb{S}^{0,n} \otimes \mathbb{S}^{0,m} \simeq \mathbb{S}^{0,n+m}.$$

As is standard by now, we conclude that this endows $\mathcal{S}\mathrm{yn}_E$ with the structure of an $\mathrm{Sp}^{\mathrm{Fil}}$-algebra in $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}$. Finally, we would like to show that it is $\mathbb{Z}$-plurigenic to apply the filtered Schwede–Shipley theorem. For this, note that the enriched mapping object $\mathrm{Map}^{\mathbb{Z}}(\mathbb{S}^{0,0},-)$ is given by the filtered spectrum

$$\mathrm{Map}^{\mathbb{Z}}(\mathbb{S}^{0,0},X)_\star = \mathrm{Map}^{\mathbb{Z}}(\mathbb{S}^{0,\star},X)$$

with structure morphisms induced by (precomposition with) τ. This follows immediately from the construction of these objects as right adjoints to the symmetric monoidal left adjoint picking out the synthetic sphere and its twists. Now note that for any t, w we have

$$\mathbb{S}^{t,w} \simeq \mathbb{S}^{t,0} \otimes \mathbb{S}^{0,w} \simeq \Sigma^t \nu\mathbb{S} \otimes \mathbb{S}^{0,w} \simeq \Sigma^t \mathbb{S}^{0,w}.$$

Since we assumed that $\mathcal{S}\mathrm{yn}_E$ was cellular by restricting our choices of E or restricting to the cellular subcategory, we see that it is generated under colimits by the $\mathbb{S}^{t,w}$ or equivalently under colimits and suspensions by the $\mathbb{S}^{0,w}$. Further noting that all these twists are dualisable ergo compact, we conclude that $\mathrm{Map}^{\mathbb{Z}}(\mathbb{S}^{0,0}, -)$ is a conservative functor and $\mathcal{S}\mathrm{yn}_E$ is $\mathbb{Z}$-plurigenic as an $\mathrm{Sp}^{\mathrm{Fil}}$-algebra. □

For completeness, we will also illustrate what happens when one expresses $\mathcal{S}\mathrm{yn}_E$ as a BHS deformation, following the sketch given in [BHS20]. We proceed to apply Definition 7.3 to the case of synthetic spectra. Note that $\mathcal{S}\mathrm{yn}_E$ is always an object of $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}})$, so that it admits an essentially unique unit map from Sp which is a symmetric monoidal left adjoint. This functor is denoted

$$c : \mathrm{Sp} \longrightarrow \mathcal{S}\mathrm{yn}_E.$$

Secondly, we need a symmetric monoidal left adjoint from $\mathcal{S}\mathrm{yn}_E$ back down to Sp. This is provided by the τ-inversion functor. Indeed, recall that the full subcategory of τ-invertible spectra was equivalent to Sp itself via the spectral Yoneda embedding. Therefore, the τ-inversion functor, which is a symmetric monoidal left adjoint, can be extended to a symmetric monoidal left adjoint

$$\mathrm{Re} : \mathcal{S}\mathrm{yn}_E \xrightarrow{\tau^{-1}} \mathcal{S}\mathrm{yn}_E[\tau^{-1}] \simeq \mathrm{Sp}.$$

Since our base category to be deformed was Sp, we see that the composite $\mathrm{Re} \circ c$ is the identity for universal reasons.

Next, we construct our graded dualisable objects, i.e. the twists of the identity. These are then given by the spheres $\mathbb{S}^{0,n}$ for $n \in \mathbb{Z}$ It is an immediate consequence of Lemma 8.7 that these all realise to the sphere spectrum $\mathbb{S} \in \mathrm{Sp}$, whence they are contained in the kernel of $\mathrm{Pic}_0\,\mathrm{Re}$. Further, this induces an equivalence on mapping spaces by Proposition 8.2, which tells us that for $n \in \mathbb{Z}$ and $k \geq 0$ we have

$$\begin{aligned} \mathrm{map}(\mathbb{S}^{0,n}, \mathbb{S}^{0,n+k}) &\simeq \mathrm{map}(\mathbb{S}^{0,0}, \mathbb{S}^{0,k}), \\ &\simeq \mathrm{map}(\nu\mathbb{S}, \Sigma^{-k}\nu\mathbb{S}^k), \\ &\simeq \Omega^{\infty+k}\nu\mathbb{S}^k(\mathbb{S}), \\ &\simeq \Omega^k\mathrm{map}(\mathbb{S}, \mathbb{S}^k), \\ &\simeq \mathrm{map}(\mathbb{S}, \mathbb{S}). \end{aligned}$$

This structure allows us to define an $\mathrm{Sp}^{\mathrm{Fil}}$-algebra structure on $\mathcal{S}\mathrm{yn}_E$ using the symmetric monoidal left adjoint induced by the symmetric monoidal functor $\mathbb{S}^{0,\star} : \mathbb{Z} \longrightarrow \mathcal{S}\mathrm{yn}_E$ as described at length in our discussion of deformations. Further, to check the analogue of the $\mathbb{Z}$-plurigenicity condition, we see that this is once again equivalent to requiring that the spheres $\mathbb{S}^{0,n}$ and their suspensions generate $\mathcal{S}\mathrm{yn}_E$ by the cellular assumption made at the beginning of this section.

Remark 9.1. It is apparent from the discussion above that synthetic spectra admit the structure of a deformation with generic fibre

$$\mathrm{GF}(\mathcal{S}\mathrm{yn}_E) \simeq \mathcal{S}\mathrm{yn}_E[\tau^{-1}] \simeq \mathrm{Sp}.$$

Now the special fibre can also be computed using the framework of [Pst18], and one obtains

$$\mathrm{SF}(\mathcal{S}\mathrm{yn}_E) \simeq \mathrm{Mod}(\mathcal{S}\mathrm{yn}_E; C\tau) \simeq \mathcal{S}\mathrm{table}_{E_*E},$$

where the latter is Hovey's stable derived category of E_*E-comodules, an ∞-categorical thickening of the derived category of the abelian category of comodules over the Hopf algebroid E_*E. Since we do not use the explicit description of the special fibre in this work, we will not elaborate on this. However, it is of great theoretical significance to the theory of synthetic spectra, since it really quantifies that statement that synthetic spectra are a deformation encapsulating both the homotopy theory of spectra (the generic fibre Sp) and the algebra of the Adams type homology theory E (the special fibre $\mathcal{S}\mathrm{table}_{E_*E}$).

Remark 9.2. As is apparent from the discussion above, since a BHS deformation requires the datum of its generic fibre in the definition, we see that we needed the construction of the realisation or τ-inversion functor (as well as the identification $\mathcal{S}\mathrm{yn}_E[\tau^{-1}] \simeq \mathrm{Sp}$ to define this deformation.

Having established this monadic adjunction, it therefore remains to identify the algebra object $\rho\mathbb{S}^{0,0}$ in filtered spectra. Unfortunately, this does not always have a simple description. Indeed, if we unravel the definition of ρ, we see that its n-th part is given by

$$\rho X_n \simeq \mathrm{Map}(\mathbb{S}^{0,n}, X)$$

for any synthetic spectrum X. This right adjoint admits a very similar construction in [Ghe+18] in the world of p-complete cellular complex motivic spectra, and in op. cit. this right adjoint is referred to as the motivic homotopy groups functor. Indeed, it is clear that the mapping spectra in the formula for ρ essentially compute the synthetic stable homotopy groups, or (equivalently) the motivic stable homotopy groups. We therefore do not expect them to have a simple description on the nose. More tractable however, is the synthetic spectrum νE and its iterated smash powers. Or more generally, synthetic analogues of spectra of the form $X \wedge E^{\wedge k}$

Proposition 9.2. *Letting ρ denote the right adjoint*

$$\rho : \mathcal{S}\mathrm{yn}_E \to \mathrm{Sp}^{\mathrm{Fil}}$$

of the deformation structure map, we can explicitly describe its value on $\nu X \otimes \nu E^k$ as

$$\rho(\nu X \otimes \nu E^{\wedge k}) \simeq \tau_{\geq \star} X \wedge E^{\wedge k},$$

where $\tau_{\geq\star} X \wedge E^{\wedge k}$ is the filtered spectrum obtained as the Whitehead tower of $X \wedge E^{\wedge k}$.

Proof. First, let us construct a comparison map for any spectrum X. Note that by the general description of enriched mapping objects in $\mathrm{Sp}^{\mathrm{Fil}}$-linear categories, we can describe the n-th component of $\rho\nu X$ as the mapping spectrum

$$\rho\nu X_n \simeq \mathrm{Map}^{\mathbb{Z}}(\mathbb{S}^{0,0}, \nu X)_n \simeq \mathrm{Map}(\mathbb{S}^{0,n}, \nu X).$$

Now the realisation functor is a left adjoint ergo exact, so that it induces a map on mapping spectra of the form

$$\mathrm{Re} : \mathrm{Map}_{\mathcal{S}\mathrm{yn}_E}(\mathbb{S}^{0,n}, \nu X) \to \mathrm{Map}_{\mathrm{Sp}}(\mathrm{Re}\,\mathbb{S}^{0,n}, \mathrm{Re}\,\nu X) \simeq \mathrm{Map}(\mathbb{S}, X) \simeq X.$$

In fact, since the structure morphisms in the filtered spectrum $\rho\nu X$ are induced by precomposition with the limit-colimit comparison map, we see that this extends to a map of filtered spectra. Indeed, we previously established that the realisation functor sends these connecting morphisms to the identity on $\mathbb{S}$, which precisely induces the structure maps in the constant filtered spectrum $\mathrm{CsMap}(\mathbb{S}, X) \simeq \mathrm{Cs}X$. We conclude that there exists a comparison map

$$\rho\nu X \to \mathrm{Cs}X$$

of filtered spectra. To show that it induces the map in our proposition, we must take a closer look at the connectivity of the source and apply the universal property of cover functors. Let us fix some index n and consider the spectrum $\rho\nu X_n$. Note that it is given by the mapping spectrum $\mathrm{Map}(\mathbb{S}^{0,n}, \nu X)$ which precisely encodes the synthetic homotopy groups of νX, i.e.

$$\pi_*\mathrm{Map}(\mathbb{S}^{0,n}, \nu X) \simeq \pi_{*,n}\nu X.$$

Now the latter homotopy groups admit an explicit description in the case that X is an E-module. In that case, one can combine Propositions 8.2 and 8.3:

- For any spectrum X with synthetic analogue νX, the synthetic homotopy groups $\pi_{*,n}\nu X$ agree with $\pi_* X$ in positive Chow degree, i.e. for $* \geq n$.
- For any homotopy E-module M with synthetic analogue νM, the synthetic homotopy groups $\pi_{*,n}\nu M$ vanish in negative Chow degree, i.e. for $* < n$.

In particular, we understand the synthetic homotopy groups of M in any range. If we let M be given by the homotopy E-module $X \wedge E^{\wedge k}$, we obtain the explicit description

$$\pi_* \mathrm{Map}(\mathbb{S}^{0,n}, \nu X \wedge E^{\wedge k}) \cong \pi_{*,n}\nu X \wedge E^{\wedge k} \cong \begin{cases} \pi_* X \wedge E^{\wedge k}, & * \geq n, \\ 0, & * < n. \end{cases}$$

Note that we liberally apply the result of Lemma 8.5 to see that there is an equivalence

$$\nu(X \wedge E^{\wedge k}) \simeq \nu X \otimes \nu E^{\wedge k}.$$

Now it is clear that the right hand side depicts the homotopy groups of the cover $\tau_{\geq n} X \wedge E^{\wedge k}$, almost per definition of the latter. Further, note that the comparison map was induced by realisation, which is precisely what gives us the equivalence above. We conclude the following:

- By inspection of the homotopy groups of the source at every filtration level, the comparison map

  $$\rho\nu X \wedge E^{\wedge k} \longrightarrow \mathrm{Cs} X \wedge E^{\wedge k}$$

  factors through the canonical cover

  $$\rho\nu X \wedge E^{k} \longrightarrow \tau_{\geq \star} X \wedge E^{\wedge k} \longrightarrow \mathrm{Cs} X \wedge E^{\wedge k}$$

  simply because the source is n-connective in filtration degree n.
- By closer inspection of the homotopy groups, we note that what survives in degree $\geq n$ is precisely isomorphic to the homotopy groups of $\tau_{\geq n} X \wedge E^{k}$, as induced by the comparison map. We therefore conclude that the factorisation above is actually an equivalence

  $$\rho\nu X \wedge E^{\wedge k} \xrightarrow{\sim} \tau_{\geq \star} X \wedge E^{\wedge k}$$

  since equivalences in filtered spectra are detected levelwise in spectra, where they are detected by homotopy groups.

□

Remark 9.3. Note that the natural filtration going between the degrees n on the synthetic homotopy groups $\pi_{*,n} X$ is precisely induced by the map

$$\tau : \mathbb{S}^{0,n-1} \longrightarrow \mathbb{S}^{0,n}.$$

Therefore, it is customary for a homotopy E-module M to write

$$\pi_{*,*}\nu M \cong \pi_* M \otimes \mathbb{Z}[\tau],$$

using our precious observations on the structure of these homotopy groups. In particular, we see that

$$\nu E_{*,*} \cong E_*[\tau], \qquad \nu E_{*,*}\nu E \cong E_* E[\tau].$$

This tells us that $(\nu E_{*,*}, \nu E_{*,*}\nu E)$ is also a flat Hopf algebroid. In fact, the yoga of Adams spectral sequences in the synthetic world can quite often be reduced to computations over a variation of the undeformed Hopf algebroid $(E_*, E_* E)$ in which a polynomial variable τ of degree one is adjoined to both. This picture is of great computational value, and also illustrates how filtrations arise naturally in the synthetic world, with this τ parameter precisely inducing the shifts.

The reason why the functor ρ does not admit a general description in all cases, is that it contains all information about synthetic spectra and their homotopy groups. Indeed, we saw from the explicit description in terms of mapping spectra that the filtered spectrum ρX associated to a synthetic spectrum X encapsulates the bigraded homotopy groups of the latter. We therefore do not expect to be able to give an explicit description of $\rho\mathbb{S}^{0,0}$ at all. However, by the above description of the values of ρ on synthetic analogues of homotopy E-modules, we can approach the synthetic sphere spectrum in a slightly more satisfactory way. The strategy is simply to resolve the synthetic spectrum $\mathbb{S}^{0,0}$ along a νE-based Adams resolution. The latter should allow us to describe the completion $\mathbb{S}^{0,0\wedge}_{\nu E}$. But by the results of the previous section, this is none other than the τ-completion $\mathbb{S}^{0,0\wedge}_{\tau}$. In conclusion, the τ-complete synthetic sphere spectrum is exactly approximated by the resolution of the synthetic sphere spectrum along the tractable synthetic analogues $(\nu E)^{\otimes k} \simeq \nu E^{\wedge k}$.

In fact, we know that τ-completion agrees with νE-completion for all synthetic analogues by Proposition 8.5. by inspection of the νE-based Adams tower. Therefore, we will be able to realise the construction sketched above for more general synthetic analogues νX.

Theorem 9.2. *Let X be a spectrum and νX its synthetic analogue. Then there is an equivalence of filtered spectra*

$$\rho(\nu X^{\wedge}_{\tau}) \simeq \mathrm{Déc}(\tau_{\geq\star}; E)(X).$$

Proof. By Proposition 8.5, we can identify the left hand side with

$$\rho(\nu X^{\wedge}_{\tau}) \simeq \rho(\nu X^{\wedge}_{\nu E}).$$

Now the synthetic spectrum in this expression is obtained as the νE-nilpotent completion of νX, i.e. the totalisation of the tensor product of νX with the cobar resolution of νE, so that

$$\rho(\nu X)^{\wedge}_{\nu E} \simeq \rho\mathrm{Tot}(\nu X \otimes \nu E^{\otimes\bullet}) \simeq \mathrm{Tot}\rho(\nu X \otimes \nu E^{\otimes\bullet}).$$

Note that we are allowed to swap the right adjoint ρ with the limit $\mathrm{Tot} = \lim_{\Delta}$. Since ν is symmetric monoidal when restricted to finite E-projective spectra in at least one variable, we can rewrite the latter as

$$\rho(\nu X \otimes \nu E^{\otimes\bullet}) \simeq \rho\nu(X \wedge E^{\wedge\bullet}) \simeq \tau_{\geq\star}X \wedge E^{\wedge\bullet},$$

where we used the key result from Proposition 9.2 If we now look at the totalisation, we obtain the desired equivalence

$$\rho(\nu X^{\wedge}_{\tau}) \simeq \mathrm{Tot}(\tau_{\geq\star}(X \wedge E^{\wedge\bullet})) =: \mathrm{Déc}(\tau_{\geq\star}; E)(X).$$

□

Remark 9.4. This result ought to be seen as an analogue to Proposition 6.8 in [Ghe+18], where Gheorghe–Isaksen–Krause–Ricka construct (everything is implicitly p-completed) a functor

$$\Gamma : \mathrm{Sp} \longrightarrow \mathrm{Sp}^{\mathrm{Fil}} : X \mapsto \mathrm{Tot}(\tau_{\geq 2\star}X \wedge \mathrm{MU}^{\wedge\bullet}),$$

and identify $\Gamma\mathbb{S}$ with $\Omega\mathbb{S}^{0,0}$, where $\mathbb{S}^{0,0}$ is the monoidal unit in $\mathrm{Sp}^{\mathrm{cell}}_{\mathbb{C}}$, and Ω is the right adjoint

$$\Omega : \mathrm{Sp}^{\mathrm{cell}}_{\mathbb{C}} \longrightarrow \mathrm{Sp}^{\mathrm{Fil}} : Y \mapsto \mathrm{Map}(S^{0,\star}, Y)$$

called the *motivic homotopy groups* functor, in analogy with our functor ρ. In fact, since there is an equivalence (still implicitly p-completed)

$$\mathrm{Sp}^{\mathrm{cell}}_{\mathbb{C}} \simeq \mathcal{S}\mathrm{yn}^{\mathrm{ev}}_{\mathrm{MU}},$$

we see that these two results are closely related. Recent work of Hahn–Raksit–Wilson in [HRW22] uses this equivalence to construct a synthetic analogue functor

$$\nu : \mathrm{Sp} \longrightarrow \mathrm{Sp}^{\mathrm{cell}}_{\mathbb{C}}$$

whose value on a bounded below $\mathbb{E}_\infty$-ring A with even complex bordism ring can be shown to be given by a similar formula

$$\nu(A) \simeq \mathrm{fil}^{\mathrm{ev}}_\star A \simeq \mathrm{Tot}(\mathrm{fil}^{\mathrm{ev}}_\star A \wedge \mathrm{MU}^{\wedge\bullet}),$$

where $\mathrm{fil}^{\mathrm{ev}}_\star$ is the even filtration on $\mathbb{E}_\infty$-rings, defined as the right Kan extension of the tower functor $\tau_{\geq 2*}$ from even $\mathbb{E}_\infty$-rings to all $\mathbb{E}_\infty$-rings. There exists a relative version of this even filtration, allowing them to consider a similar formula as above for module M relative to an $\mathbb{E}_\infty$-ring A.

Remark 9.5. Although one really can not get rid of the τ-completion, the latter being essential to allow us to approximate a synthetic analogue by its Adams filtration, we do recall that τ-completeness of νX is equivalent to νE-nilpotent completeness of νX, which is further equivalent to E-nilpotent completeness of X. While the latter assumption is not simpler, we do know for example that any connective spectrum is MU-nilpotent complete. This is a classical result central to the study of the Adams–Novikov spectral sequence, as reviewed in [Rav84]. In particular, since the sphere spectrum is connective, we see that in MU-synthetic spectra

$$\rho\nu\mathbb{S} \simeq \rho\nu\mathbb{S}^\wedge_{\mathrm{MU}} \simeq \rho\nu\mathbb{S}^\wedge_{\nu\mathrm{MU}} \simeq \rho\nu\mathbb{S}^\wedge_\tau \simeq \mathrm{Déc}(\tau_{\geq\star};\mathrm{MU})(\mathbb{S}).$$

At this point, we must warn the reader than one primarily cares about even synthetic spectra based on MU for their comparison with cellular complex motivic spectra. In that case, we only need to add the (in this case rather obviously true) assumption that $\mathrm{MU}_*\mathbb{S}$ be even, so that its image under the even synthetic analogue is modeled by the filtered spectrum

$$\mathrm{Déc}(\tau_{\geq 2\star};\mathrm{MU})(\mathbb{S})$$

that appears in [Ghe+18]. This remark about MU-nilpotent completeness of connective (or even bounded below) spectra ergo τ-completeness of their synthetic analogues, along with the comment above about even gradings is precisely the condition that appears in Hahn–Raksit–Wilson's formula in 9.4.

As previously stated in the motivation for this discussion, we see that one can plug in $X = \mathbb{S}$ to obtain a description of the τ-complete unit in synthetic spectra. In particular, this gives us an explicit formula for τ-complete synthetic spectra in terms of filtered spectra.

Theorem 9.3. *The subcategory of τ-complete synthetic spectra $\mathcal{S}\mathrm{yn}_E{}^\wedge_\tau$ can be described as*

$$\mathcal{S}\mathrm{yn}_E{}^\wedge_\tau \simeq \mathrm{Mod}(\mathrm{Sp}^{\mathrm{Fil}}{}^\wedge_\tau;\mathrm{Déc}(\tau_{\geq\star};E)(\mathbb{S})),$$

where the algebra object in the latter is the décalage defined in 5.1.

Proof. This is a rather immediate consequence of the discussion above. Indeed, by the $\mathbb{Z}$-plurigenicity result for the deformation structure on synthetic spectra, we had an equivalence

$$\mathcal{S}\mathrm{yn}_E \simeq \mathrm{Mod}(\mathrm{Sp}^{\mathrm{Fil}};\rho\mathbb{S}^{0,0}).$$

Now note that the adjunction $\lambda \dashv \rho$ inducing this equivalence preserves the deformation parameter τ on both sides. Indeed, it is clear that

$$\lambda(\tau) : \lambda\mathbb{1}(-1) = \mathbb{S}^{0,-1} \to \lambda\mathbb{1} = \mathbb{S}^{0,0}$$

is the τ operator from synthetic spectra, and conversely

$$\rho(\tau) : \rho\mathbb{S}^{0,-1} = \mathrm{Map}(\mathbb{S}^{0,\star},\mathbb{S}^{0,-1}) \to \rho\mathbb{S}^{0,0}\mathrm{Map}(\mathbb{S}^{0,\star},\mathbb{S}^{0,0})$$

is the shift in filtered spectra by the identification

$$\mathrm{Map}(S^{0,\star},S^{0,-1}) \simeq \mathrm{Map}(S^{0,\star+1},S^{0,0}).$$

We conclude that the equivalence between synthetic spectra and modules in filtered spectra restricts to an equivalence on τ-complete subcategories on both sides. In particular, we obtain

$$\mathcal{S}\mathrm{yn}_E{}^\wedge_\tau \simeq \mathrm{Mod}(\mathrm{Sp}^{\mathrm{Fil}}{}^\wedge_\tau;\rho\mathbb{S}^{0,0}{}^\wedge_\tau) \simeq \mathrm{Mod}(\mathrm{Sp}^{\mathrm{Fil}}{}^\wedge_\tau;\mathrm{Déc}(\tau_{\geq\star};E)(\mathbb{S})),$$

where the last line uses our previous computation of the synthetic homotopy groups of τ-complete synthetic analogues. $\square$

Remark 9.6. Recall that there was a symmetric monoidal equivalence $\mathcal{K}(\mathrm{Sp}) \simeq \mathrm{Sp}^{\mathrm{Fil}\wedge}_{\tau}$. In particular, this allows us to describe τ-complete synthetic spectra not just in terms of a filtered model but even a cochain complex model. The equivalence was given in one direction by

$$\lambda : \mathrm{Sp}^{\mathrm{Fil}\wedge}_{\tau} \to \mathcal{K}(\mathrm{Sp}) : X \mapsto \Sigma^{*}\mathrm{gr}_{*}X.$$

Specifically, one can compute its value on $\mathrm{D\acute{e}c}(\tau_{\geq\star}; E)(\mathbb{S})$ using

$$\begin{aligned}
\mathrm{gr}_n \mathrm{D\acute{e}c}(\tau_{\geq\star}; E)(\mathbb{S}) &\simeq \mathrm{D\acute{e}c}(\tau_{\geq\star}; E)(\mathbb{S})_n / \mathrm{D\acute{e}c}(\tau_{\geq\star}; E)(\mathbb{S})_{n+1}, \\
&\simeq \mathrm{cof}(\mathrm{Tot}(\tau_{\geq n+1}E^{\wedge\bullet}) \to \mathrm{Tot}(\tau_{\geq n}E^{\wedge\bullet})), \\
&\simeq \mathrm{Tot}(\mathrm{cof}(\tau_{\geq n+1}E^{\wedge\bullet} \to \tau_{\geq n}E^{\wedge\bullet})), \\
&\simeq \mathrm{Tot}(\Sigma^{n} H\pi_n E^{\wedge\bullet}).
\end{aligned}$$

Indeed, by stability we can commute the limit Tot with the cofibre sequence, We then note that the associated graded of the Whitehead tower is given by (shifts of (Eilenberg–MacLane spectra on)) its homotopy groups. These assemble to a cochain complex in spectra of the form

$$\lambda \mathrm{D\acute{e}c}(\tau_{\geq\star}; E)(\mathbb{S}) \simeq \Sigma^{*}\mathrm{Tot}(\Sigma^{n} H\pi_n E^{\wedge\bullet}).$$

We conclude that there is a further equivalence

$$\mathcal{S}\mathrm{yn}_{E}{}^{\wedge}_{\tau} \simeq \mathrm{Mod}(\mathcal{K}(\mathrm{Sp}); \Sigma^{*}\mathrm{Tot}(\Sigma^{n} H\pi_n E^{\wedge\bullet})).$$

Now that we have stated the main result of our discussion of synthetic spectra as a deformation, namely an identification of the τ-complete subcategory using an explicit filtered model, let us quantify how much of this can be worked back up to describe all of $\mathcal{S}\mathrm{yn}_E$. For this, we will use our discussion of recollements to note that the deformation structure on $\mathcal{S}\mathrm{yn}_E$ induces a recollement into the τ-invertible and τ-complete parts of $\mathcal{S}\mathrm{yn}_E$, of the form

$$\begin{tikzcd}
\mathcal{S}\mathrm{yn}_E[\tau^{-1}] \arrow[r] & \mathcal{S}\mathrm{yn}_E \arrow[r] \arrow[l, bend right] \arrow[l, bend left] & \mathcal{S}\mathrm{yn}_{E}{}^{\wedge}_{\tau}. \arrow[l, bend right] \arrow[l, bend left]
\end{tikzcd}$$

By the results of this section and the previous one, we can identify

$$\mathcal{S}\mathrm{yn}_E[\tau^{-1}] \simeq \mathrm{Sp}, \qquad \mathcal{S}\mathrm{yn}_{E}{}^{\wedge}_{\tau} \simeq \mathrm{Mod}(\mathrm{Sp}^{\mathrm{Fil}\wedge}_{\tau}; \mathrm{D\acute{e}c}(\tau_{\geq\star}; E)(\mathbb{S})).$$

By our discussion of recollements associated to a dualisable homotopy associated algebra object, we see that this recollement is symmetric monoidal. Further, recall that the reconstruction formula for recollements in terms of lax limits gives a symmetric monoidal equivalence. This gives us the final result

Corollary 9.2. *There is a symmetric monoidal equivalence of ∞-categories*

$$\mathcal{S}\mathrm{yn}_E \simeq \mathrm{Mod}(\mathrm{Sp}^{\mathrm{Fil}\wedge}_{\tau}; \mathrm{D\acute{e}c}(\tau_{\geq\star}; E)(\mathbb{S})) \times_{\tau^{-1},\mathrm{Sp},t} \mathrm{Sp}^{\Delta^1}.$$

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