Rigid analytic coordinates on the stack of one-dimensional formal groups

Laurent Fargues

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Abstract

Following the founding work of Quillen one can relate the stable homotopy category of spectra to quasicoherent sheaves on the stack of 1-dimensional formal groups via Lazard's ring. This point of view has given birth to chromatic homotopy theory. In this article we present some new analytic rigid coordinates on this stack that are not known from specialists of chromatic homotopy theory, coordinates that could be usefull [sic.] when coupled with the recent advances on quasicoherent sheaves in rigid analytic geometry.

Translated from French by Lucas Piessevaux. Send any comments and typos to lucas@math.uni-bonn.de

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0 Introduction

Following Quillen, we can associate to a CW complex X a quasicoherent sheaf $\mathrm{MU}_*(X)$ on the stack of one-dimensional formal groups. The stratification of this stack by the height of formal group laws at a prime number

p then allows us to construct cohomology theories parametrised by p-local spectra that interpolate between K-theory and usual mod p cohomology. We can therefore associate to any spectrum X a tower of localisations

$$X \to \cdots \to L_{E(n)}X \to L_{E(n-1)}X \to \cdots \to L_{E(0)}X$$

whose homotopy limit is X when X is p-local and finite (the chromatic convergence theorem). This provides a resolution of X by homotopy fibres of $L_nX \to L_{n-1}X$ as n varies, these fibres being related to Lubin–Tate spaces.

Within this framework, all associated computations rely on the description of the p-typical Lazard ring given by Cartier theory of formal groups

$$\mathbb{Z}_{(p)}[v_i]_{i\geq 1}.$$

In this text, we propose a different point of view, not algebraic but rigid analytic, as in [Far04] where the motto was to do everything *on the generic fibre*, without using integral models as in [HT01]. It is this point of view that finally lead to [FS21], where the stack Bun_G of G-bundles on the curve is of rigid analytic nature, as opposed to that of F-iscocrystals. In this perspective, the *crystalline* point of view of Cartier theory, as appearing in [HG94], is not as well adapted as the point of view of *Hodge–Tate periods*. In fact, there are two applications of periods in the framework of p-adic geometry:

- the de Rham periods, considered in [HG94] and [RZ96, Chapter 5] associated to the Hodge–de Rham spectral sequence,
- the Hodge–Tate periods associated to the Hodge–Tate spectral sequence.

These are reflected in the two morphisms

$$\mathrm{Bun}_G \longleftarrow \mathrm{Hecke} \longrightarrow \mathrm{Bun}_G$$

of [FS21] defining the Hecke stack of modifications of bundles on the curve. The perspective of Hodge–Tate periods has appeared in [Fal02] and [FGL08]. It has recently been exploited in [Bar+24], which exploits the isomorphism of [FGL08]. This isomorphism can be reformulated in modern language as an isomorphism of v-stacks

$$[\underline{\mathcal{O}_D^\times}\backslash \mathrm{LT}_\eta^\diamond] \simeq [\underline{\mathrm{GL}_n(\mathbb{Z}_p)}\backslash \Omega^\diamond]$$

where LT is the Lubin–Tate space with generic fibre LT_η, a rigid analytic open ball, and Ω is a Drinfeld space.

In this text, we propose to extend this point of view by proving the following result

Theorem 0.1. *Let $\widehat{\mathcal{M}}_{\mathrm{FG}}^{\leq h}$ be the stack over $\mathrm{Spf}(\mathbb{Z}_p)$ of formal groups of dimension 1 and height $\leq h$. One can define the diamond associated to its overconvergent generic fibre*

$$\widehat{\mathcal{M}}_{\mathrm{FG},p,\eta}^{\leq h,\diamond,\dagger}$$

over $\mathrm{Spa}(\mathbb{Q}_p)^\diamond$.

1. *There is a surjective morphism of v-stacks*

$$\begin{array}{c}[\underline{\mathrm{GL}_h(\mathbb{Z}_p)}\backslash \mathbb{P}_{\mathbb{Q}_p}^{h-1,\diamond}] \\ \big\downarrow \\ \widehat{\mathcal{M}}_{\mathrm{FG},p,\eta}^{\leq h,\diamond,\dagger}\end{array}$$

given by the Hodge–Tate periods.

2. *We can explicitly describe the associated presentation of the stack $\widehat{\mathcal{M}}^{\leq h,\diamond,\dagger}_{\mathrm{FG},p,\eta}$, i.e. the groupoid*

$$\begin{array}{c} \mathbb{P}^{h-1,\diamond}_{\mathbb{Q}_p} \times_{\widehat{\mathcal{M}}^{\leq h,\diamond,\dagger}_{\mathrm{FG},p,\eta}} \mathbb{P}^{h-1,\diamond}_{\mathbb{Q}_p} \\ \downarrow \uparrow \downarrow \\ \mathbb{P}^{h-1,\diamond}_{\mathbb{Q}_p}. \end{array}$$

3. *The height stratification of $\widehat{\mathcal{M}}^{\leq h,\diamond,\dagger}_{\mathrm{FG},p,\eta}$ corresponds to the stratification of $|\mathbb{P}^{h-1}_{\mathbb{Q}_p}|$ given by, for $K/\mathbb{Q}_p$, the dimension of the kernel of a nonzero $\mathbb{Q}_p$-linear morphism $\mathbb{Q}_p^h \to K$. In particular, the open stratum is the Drinfeld space Ω.*

The idea is thus to replace the coordinates

$$(v_1, v_2, \ldots)$$

coming from Cartier theory by coordinates

$$[x_0 : \cdots : x_{h-1}]$$

on the projective space $\mathbb{P}^{h-1}_{\mathbb{Q}_p}$. We hope that this point of view is useful. It consists of a simple, probably rather naïve, suggestion–the author not being an expert in the subject–but probably worthy of note.

The relevance of such a theorem is that one nevertheless has a *good* notion of quasicoherent coefficients on such objects.

Corollary 0.2. *One can construct a natural functor*[1]

$$\text{Quasicoherent sheaves on } \widehat{\mathcal{M}}^{\leq h}_{\mathrm{FG},p} \to \underline{\mathrm{GL}_h(\mathbb{Z}_p)}\text{-equivariant quasicoherent complexes à la Mann on } \mathbb{P}^{h-1,\diamond}_{\mathbb{Q}_p}.$$

Here, the quasicoherent complexes are those defined by Mann in [Man22], cf. also [AM24]. They consists of complexes of $\mathcal{O}^{\sharp,+,a}$-modules which are p-adically complete (in the derived sense). Pro-étale locally on $\mathbb{P}^{h-1,\mathrm{ad}}_{\mathbb{Q}_p}$, on an affinoid perfectoid chart of the form $\mathrm{Spa}(A, A^+) \to \mathbb{P}^{h-1,\mathrm{ad}}_{\mathbb{Q}_p}$ with A of *finite type*, (cf. [Man22, Proposition 3.1.9]) these are given by the objects of the stable ∞-category

$$\{C \in \mathcal{D}^b(A^+, a) \mid C \xrightarrow{\sim} \varprojlim_{n\geq 1} C/p^n\}.$$

Henceforth, we obtain a functor

$$p\text{-complete spectra} \to \underline{\mathrm{GL}_h(\mathbb{Z}_p)}\text{-equivariant quasicoherent complexes } \textit{à la Mann} \text{ on } \mathbb{P}^{h-1,\diamond}_{\mathbb{Q}_p}.$$

Upon inverting p, given recent progress in p-adic Simpson theory (the reformulation of work of Faltings ([Fal05], [Fal11], [AGT16]) in *modern* terms, cf. for example [Heu23], [AHB23]) extended to perfect complexes one could hope for a construction of the form

$$p\text{-local finite spectra}[\frac{1}{p}] \dashrightarrow \mathrm{GL}_h(\mathbb{Z}_p)\text{-equivariant perfect Higgs complexes on } \mathbb{P}^{h-1}_{\mathbb{Q}_p},$$

cf. section 7. From the point of view in [HG94] on equivariant bundles on the Lubin–Tate space, this consists of viewing $\mathrm{GL}_h(\mathbb{Z}_p)$-equivariant Higgs bundle on Drinfeld's space Ω_{h-1} of dimension $h-1$. One of the fascinating aspects of such a construction would be that, according to GAGA, it would produce perfect complexes of algebraic Higgs bundles on projective space by means of a rigid analytic detour.

Historical note: The author of this work tried to disseminate these ideas since 2008 amidst the chromatic homotopy theory community, such as Jack Morava or Michael Hopkins ([Far], based on talks about [FGL08]. It was probably early [2] *and the apparition of [Bar+24] along with the insistence of Lars Hesselholt have incited the author to write up this note.*

[1] Translator's note: Fargues writes $\mathrm{GL}_n(\mathbb{Z}_p)$–presumably meaning $\mathrm{GL}_h(\mathbb{Z}_p)$

[2] Translator's note: Fargues writes *tôtnd smith* [sic.]. Not sure what this means.

1 The stack of formal groups of dimension one

1.1 Generalities

1.1.1 The stack over $\mathrm{Spec}(\mathbb{Z})$

Definition 1.1. Let us denote by $\mathcal{M}_{\mathrm{FG}}$ the fpqc stack that associates to a scheme S the groupoid of commutative formal Lie S-groups of dimension 1.

In the preceding definition a *formal Lie S-group* is formal S-scheme in groups, formally of finite type and formally smooth, i.e. locally on S isomorphic to $\widehat{\mathbb{A}}^d_S$ (with $d = 1$ in our case). We refer to [Mes06, Chapter II] for generalities on formal Lie groups. Let us consider the Zariski presentation

$$\widetilde{\mathcal{M}}_{\mathrm{FG}} \to \mathcal{M}_{\mathrm{FG}}$$

given by the moduli space of *one-dimensional formal group laws*. In other words, $\widetilde{\mathcal{M}}_{\mathrm{FG}}(S)$ is the set of pairs $(\mathcal{G}, \iota)$, where $\mathcal{G} \in \mathcal{M}_{\mathrm{FG}}(S)$ and $\iota\colon \widehat{\mathbb{A}}^1_S \xrightarrow{\sim} \mathcal{G}$ is an isomorphism of pointed formal S-schemes (the zero section is sent to the neutral section by ι). We have

$$\widetilde{\mathcal{M}}_{\mathrm{FG}} = \mathrm{Spec}(\Lambda),$$

where $\Lambda \simeq \mathbb{Z}[t_k]_{k\geq 1}$ is the Lazard ring. Hence,

$$\mathcal{M}_{\mathrm{FG}} = [H \diagdown \widetilde{\mathcal{M}}_{\mathrm{FG}}]$$

(the Zariski quotient) with

$$H = \underline{\mathrm{Aut}}(\widehat{\mathbb{A}}^1)$$

(automorphisms preserving the origin), a pro-unipotent $\mathbb{Z}$-group scheme acting on $\widetilde{\mathcal{M}}_{\mathrm{FG}}$.

1.1.2 Localisation at p

Let us now fix a prime number p. We then obtain a presentation of

$$\mathcal{M}_{\mathrm{FG},(p)} := \mathcal{M}_{\mathrm{FG}} \otimes \mathbb{Z}_{(p)}$$

given by the moduli space of p-typical formal group laws; the pointed isomorphisms $\iota\colon \widehat{\mathbb{A}}^1_S \xrightarrow{\sim} \mathcal{G}$ such that the associated curve is p-typical, i.e. for all $n \geq 1$ such that $(n, p) = 1$ we have $F_n\iota = 0$ in the Cartier module ([Zin]) . Here, if

$$\gamma\colon \widehat{\mathbb{A}}^1_S \to \mathcal{G}$$

is a curve, i.e. a morphism of formal pointed S-schemes, then

$$F_n\gamma = \mathrm{tr}_{\phi_n}\gamma.$$

The morphism $\phi_n\colon \widehat{\mathbb{A}}^1_S \to \widehat{\mathbb{A}}^1_S$ is given by $\phi_n^* T = T^n$, T being the coordinate on $\widehat{\mathbb{A}}^1_S$, and

$$\mathrm{tr}_{\phi_n}\colon \phi_{n*}\phi_n^*\mathcal{G}_{\widehat{\mathbb{A}}^1_S} \to \mathcal{G}_{\widehat{\mathbb{A}}^1_S}$$

is the trace map ([BD06, pp. XVII–6.3.4]). This moduli space

$$\widetilde{\mathcal{M}}_{\mathrm{FG},(p)}$$

is isomorphic to

$$\mathrm{Spec}(\mathbb{Z}_{(p)}[v_k]_{k\geq 1}).$$

The universal p-typical Cartier module is generated by the curve γ subject to the relation

$$F\gamma = \sum_{k\geq 0} V^k[v_{k+1}]\gamma$$

in the associated Cartier module. In other words, the associated formal group law has logarithm given by the formal power series $f \in \mathbb{Z}_{(p)}[\![T]\!]$ satisfying $f(0) = 0, f'(0) = 1$ and the functional equation

$$f(T) = T + \sum_{k \geq 1} \frac{f(v_k T^{p^k})}{p}.$$

Consider the closed subgroupoid

$$\mathcal{N} \subset H \times_{\mathrm{Spec}(\mathbb{Z})} \widetilde{\mathcal{M}}_{\mathrm{FG},(p)}$$

of $H \times_{\mathrm{Spec}(\mathbb{Z})} \widetilde{\mathcal{M}}_{\mathrm{FG},(p)}$ acting on $\widetilde{\mathcal{M}}_{\mathrm{FG},(p)}$ consisting of elements $(h, \mathcal{G}, \iota)$ where $h \in H$ and $(\mathcal{G}, \iota) \in \widetilde{\mathcal{M}}_{\mathrm{FG},(p)}$ are such that $\iota \circ h$ is still p-typical. If we denote $\widetilde{\mathcal{M}}_{\mathrm{FG},(p)} = \mathrm{Spec}(R)$,

$$X \underset{\mathfrak{F}}{+} Y = f^{-1}(f(X) + f(Y))$$

and $\underset{\mathfrak{F}}{+}$ is the universal formal group law with coefficients on R, we can describe it for any R-algebra A by

$$\mathcal{N}(A) = \{T + {\sum_{k\geq 1}}^{\mathfrak{F}} a_k T^{p^k} \mid a_k \in A\} \subset T + T^2 A[\![T]\!] = H(A).$$

where the notation $\sum^{\mathfrak{F}}$ denotes taking the sum for the formal group law $\mathfrak{F}$. We then have

$$\mathcal{M}_{FG,(p)} = \mathrm{coeq}(\mathcal{N} \rightrightarrows \widetilde{\mathcal{M}}_{\mathrm{FG},(p)})$$

(the stacky Zariski quotient).

1.1.3 Completion at p and height stratification

Definition 1.2. Denote by $\widehat{\mathcal{M}}_{\mathrm{FG},p}$ the stack over $\mathrm{Spf}(\mathbb{Z}_p)$ obtained by p-adic completion of $\mathcal{M}_{\mathrm{FG}}$.

By stack over $\mathrm{Spf}(\mathbb{Z}_p)$ we lean that the test objects are schemes on which p is locally nilpotent, the topology is the fpqc topology, and the value on $S \in \mathrm{Nilp}_{\mathbb{Z}_p}$ is the groupoid of one-dimensional formal Lie groups over S. There is a stratification by the height $h \in \mathbb{N}_{\geq 1} \cup \{+\infty\}$ of the formal group such that

$$\widehat{\mathcal{M}}_{\mathrm{FG},p}^{\geq h}$$

is a Zariski closed substack. We have

$$|\widehat{\mathcal{M}}_{\mathrm{FG},p}| = |\widehat{\mathcal{M}}_{\mathrm{FG},p,\mathrm{red}}| = \mathbb{N}_{\geq 1} \cup \{+\infty\}$$

with the topology given by the opposite of the usual order on $\mathbb{N}_{\geq 1} \cup \{+\infty\}$.

Using the coordinates $(v_k)_{k\geq 1}$ of the previous section, we have

$$\widehat{\mathcal{M}}_{\mathrm{FG},p}^{\geq h} = \{p = v_1 = \cdots = v_{h-1} = 0\}.$$

1.1.4 Formal completion at a point of finite height.

Let us fix $h \geq 1$. Let $\mathcal{G}_h$ be the p-divisible formal group of height h over $\mathbb{F}_p$ with covariant crystal $\mathbb{Z}_p^h$ equipped with the Verschiebung given by the matrix

$$\begin{pmatrix} 0 & \cdots & \cdots & p \\ 1 & & & \\ & \ddots & & \\ & & 1 & 0 \end{pmatrix}.$$

We associate to it the point

$$x_h \in \widehat{\mathcal{M}}_{\mathrm{FG},p}(\mathbb{F}_p).$$

Definition 1.3. 1. Let us denote by

$$\widehat{\mathcal{M}}^{(h)}_{\mathrm{FG},p}$$

the formal completion of $\widehat{\mathcal{M}}_{\mathrm{FG},p}$ along x_h[3].

2. Denote by

$$\mathfrak{X}_h = \mathrm{Def}(\mathcal{G}_h)$$

the Lubin–Tate space of deformations of $\mathcal{G}_h$.

3. Denote by J_h the pro-étale sheaf of groups on schemes locally annihilated by a power of p given by

$$J_h(S) = \mathrm{Aut}(\mathcal{G}_h \times_{\mathrm{Spec}(\mathbb{F}_p)} S).$$

To be precise, the Lubin–Tate space is the extension of scalars from $\mathrm{Spf}(\mathbb{Z}_p)$ to $\mathrm{Spf}(W(\overline{\mathbb{F}}_p))$ of this aforementioned space. We have

$$\mathfrak{X}_h \simeq \mathrm{Spf}(\mathbb{Z}_p[\![x_1, \ldots, x_{h-1}]\!]).$$

If $D_h = \mathbb{Q}_{p^h}[\Pi]$ is the division algebra of invariant $\frac{1}{h}$ over $\mathbb{Q}_p$, J_h is étale-locally constant isomorphic to $\underline{D_h^\times}$. More precisely

$$J_h \times_{\mathrm{Spec}(\mathbb{F}_p)} \mathrm{Spec}(\mathbb{F}_{p^h}) = \underline{\mathcal{O}_{D_h}^\times}.$$

The following result is then classical.

Proposition 1.4. *There is then an isomorphism*

$$\widehat{\mathcal{M}}^{(h)}_{\mathrm{FG},p} \xrightarrow{\sim} [J_h \diagdown \mathfrak{X}_h]$$

where the stacky quotient is taken for the pro-étale topology on schemes locally annihilated by a power of p.

The pro-étale J_h-torsor

$$\mathfrak{X}_h \to \widehat{\mathcal{M}}^{(h)}_{\mathrm{FG},p}$$

is the *Igusa torsor* ([HT01, Chapter IV]).

2 The generic fibre

We refer the reader to [Sch17, Section 8] for the v-topology on perfectoid spaces. In the following definition, the symbol η denotes the generic fibre of our formal stack. We do not make sense of this generic fibre, but can make sense of its diamond as a v-stack. Let us note furthermore that we take the *overconvergent* version. If $\mathfrak{X}$ is a separated formal $\mathrm{Spf}(\mathbb{Z}_p)$-scheme locally formally of finite type[4], then $\mathfrak{X}_\eta^{\diamond,\dagger}$ is none other than the diamond of Huber's canonical compactfication of the generic fibre $\mathfrak{X}_\eta$. For example, if $\mathfrak{X} = \mathrm{Spf}(\mathbb{Z}_p\langle T\rangle)$ then

$$\mathfrak{X}_\eta^{\diamond,\dagger} = \mathrm{Spa}(\mathbb{Q}_p\langle T\rangle, \mathbb{Z}_p + pT\mathbb{Z}_p\langle T\rangle)^\diamond,$$

Huber's canonical compactification of the *classical* rigid analytic open ball.

Definition 2.1. Let us denote by

$$\widehat{\mathcal{M}}^{\diamond,\dagger}_{\mathrm{FG},p,\eta}$$

the v-stack on the category of perfectoid spaces over $\mathbb{F}_p$ which is the stack associated to the prestack that associates to a affinoid perfectoid $\mathbb{F}_p$-algebra (R, R^+) an untilt $(R^\sharp, R^{\sharp,+})$ over $\mathbb{Q}_p$ and an object of $\widehat{\mathcal{M}}_{\mathrm{FG},p}(R^{\sharp,\circ})$.

[3] Translator's note: Fargues writes the completion of $\mathcal{M}_{\mathrm{FG},p}$ [sic.], presumably meaning $\widehat{\mathcal{M}}_{\mathrm{FG},p}$

[4] Translator's note: Fargues writes *un* $\mathrm{Spf}(\mathbb{Z}_p)$*-schéma formel séparé local séparé localement formellement de type fini* [sic.], we are not sure how to parse this correctly and may have made a mistake in the assumptions on $\mathfrak{X}$.

We similarly define

$$\widehat{\mathcal{M}}^{\leq h, \diamond, \dagger}_{\mathrm{FG}, p, \eta}$$

which is the object in which we are interested as h varies, i.e. we are only interested in the finite height part.

We will also be interested in the stack of p-divisible groups of a fixed height h. We denote by

$$\mathrm{BT}_h$$

the stack of p-divisible groups of height h over $\mathrm{Spec}(\mathbb{Z}_p)$, and $\widehat{\mathrm{BT}}_h$ its formal completion over $\mathrm{Spf}(\mathbb{Z}_p)$ obtained by restriction to p-divisible groups on schemes locally annihilated by p.

Definition 2.2. For $h \geq 1$ and $g \geq 0$ we denote by

$$\widehat{\mathrm{BT}}^{\diamond, \dagger}_{d, h, \eta}$$

the v-stack associated to the prestack that associated to an affinoid perfectoid $\mathbb{F}_p$-algebra (R, R^+) an untilt $(R^\sharp, R^{\sharp,+})$ over $\mathbb{Q}_p$ and a p-divisible group of dimension d and height h over $\mathrm{Spf}(R^{\sharp,\circ})$[5]

There is an evident morphism

$$\widehat{\mathrm{BT}}_h \to \widehat{\mathcal{M}}^{\leq h}_{\mathrm{FG}, p}$$

given by the operation of formal completion of a p-divisible group ([Mes06, Chapter III]).

Definition 2.3. We denote by

$$\pi_h \colon \widehat{\mathrm{BT}}^{\diamond, \dagger}_{1, h, \eta} \to \widehat{\mathcal{M}}^{\leq h, \diamond, \dagger}_{\mathrm{FG}, p, \eta}$$

the morphism induced by the formal completion of a p-divisible group.

Let us begin with a lemma that justifies the introduction of this morphism.

Lemma 2.4. *The morphism π_h is v-surjective.*

Proof. Let $S = \mathrm{Spa}(R, R^+)$ be an affinoid perfectoid space over $\mathbb{Q}_p$ and $\mathcal{G}$ a one-dimensional formal Lie group on $\mathrm{Spf}(R^\circ)$ of height $\leq h$. For all $s \in S$ let us consider the formal Lie group

$$\mathcal{G} \widehat{\otimes}_{R^\circ} K(s)^\circ.$$

Since $K(s)^\circ$ is a valuation ring of height 1, the height of this formal group over $\mathrm{Spf}(K(s)^\circ)$ is constant, whence it consists of a p-divisible group of height $t(s) \leq h$. Let us then denote

$$H_s = \mathcal{G} \widehat{\otimes}_{R^\circ} K(s)^\circ \oplus (\mathbb{Q}_p / \mathbb{Z}_p)^{h - t(s)}$$

which is p-divisible of height h.

We then consider the perfectoid $\mathbb{Q}_p$-algebra (A, A^+) with

$$A^+ = \prod_{s \in S} K(s)^+$$

and $A = A^+[\frac{1}{\varpi}]$, where ϖ is a pseudouniformiser of R. The morphism

$$\mathrm{Spa}(A, A^+) \to S$$

is obviously v-surjective. Furthermore, we have

$$A^\circ = \prod_{s \in S} K(s)^\circ.$$

[5] Translator's note: Fargues writes that this p-divisible group should live over $\mathrm{Spf}(R^\circ)$. We assume this is a typo and it should live over $\mathrm{Spf}(R^{\sharp,\circ})$.

For any integer $n \geq 1$ we consider the A°-algebra

$$B_n = \prod_{s\in S} \mathcal{O}(H_s[p^n]).$$

The following Lemma 2.5 shows that $\varinjlim_{n\geq 1} \mathrm{Spec}(B_n)$ is a p-divisible group. Let us denote by $\mathcal{O}(H_s[p^n])^+$ the augmentation ideal associated to the unit section. We then have

$$B_k^+ = \prod_{s\in S} \mathcal{O}(H_s[p^n])^+$$

and therefore for any $k \geq 1$

$$(B_n^+)^k = \prod_{s\in S} (\mathcal{O}(H_s[p^n])^+)^k.$$

The formal completion of our p-divisible group is therefore given by the algebra

$$\varprojlim_{n\geq 1, k\geq 1} \prod_{s\in S} [\mathcal{O}(H_s[p^n])/(\mathcal{O}(H_s[p^n])^+)^k]$$

which is hence given by

$$\varprojlim_{n\geq 1, k\geq 1} \prod_{s\in S} \mathcal{O}(\mathcal{G}\widehat{\otimes}_{R^\circ} K(s)^\circ[p^n])/(\mathcal{O}(\mathcal{G}\widehat{\otimes}_{R^\circ} K(s)^\circ[p^n])^+)^k,$$

which is furthermore given by

$$\prod_{s\in S} \mathcal{O}(\mathcal{G}\widehat{\otimes}_{R^\circ} K(s)^\circ).$$

It therefore suffices to see that this product can be identified with $\mathcal{O}(\mathcal{G})\widehat{\otimes}_{R^\circ} \prod_{s\in S} K(s)^\circ$. This is clear by writing $\mathcal{O}(\mathcal{G}) \simeq R^\circ[\![T]\!]$. □

Lemma 2.5. *Let $h \geq 1$ be an integer.*

1. *Let $(A_i)_{i\in I}$ be a collection of rings and for all $i \in I$, H_i a p-divisible group over* $\mathrm{Spec}(A_i)$ *of height h such that for all integers $n \geq 1$, $\mathcal{O}(H_i[p^n])$ is a free A_i-module. Then the collection*

$$(\mathrm{Spec}(\prod_{i\in I} \mathcal{O}(H_i[p^n])))_{n\geq 1}$$

   is a p-divisible group of height h over $\mathrm{Spec}(\prod_{i\in I} A_i)$.

2. *If furthermore $(H_i')_{i\in I}$ is another collection of p-divisible groups satisfying the same hypotheses as $(H_i)_{i\in I}$, then*

$$\mathrm{Hom}(\varinjlim_{n\geq 1} \mathrm{Spec}(\prod_{i\in I} \mathcal{O}(H_i[p^n])), \varinjlim_{n\geq 1} \mathrm{Spec}(\prod_{i\in I} \mathcal{O}(H_i'[p^n]))) = \prod_{i\in I} \mathrm{Hom}(H_i, H_i').$$

3 The generic fibre of the stack of Barsotti–Tate groups.

Let us now identify $\widehat{\mathrm{BT}}^{\diamond,\dagger}_{1,H,\eta}$. The v-stack

$$\underline{\mathrm{GL}_h(\mathbb{Z}_p)}\diagdown\mathbb{P}^{h-1,\diamond}_{\mathbb{Q}_p}$$

associates to a perfectoid space S over $\mathbb{F}_p$ an untilt $S^\sharp$, a pro-étale $\mathbb{Q}_p$-sheaf $\mathcal{F}$ locally isomorphic to $\underline{\mathbb{Q}_p^h}$, and a surjective morphism

$$\mathcal{F} \otimes_{\underline{\mathbb{Q}_p}} \mathcal{O}_{S^\sharp} \to \mathcal{L}$$

where $\mathcal{L}$ is a locally free rank one $\mathcal{O}_{S^\sharp}$-module.

If $S = \mathrm{Spa}(A, A^+)$ is an affinoid perfectoid space over $\mathbb{Q}_p$ and H is a p-divisible group of height h in A°, we can consider the pro-étale $\underline{\mathrm{GL}_h(\mathbb{Z}_p)}$-torsor associated to the Tate module of $H^D \otimes_{A^\circ} A$. Above this torsor $T = \mathrm{Spa}(B, B^+)$, there is a morphism

$$\alpha_{H^D} : \underline{\mathbb{Q}_p^h} \to \omega_H[\frac{1}{p}] \otimes_A B.$$

This allows us to define a Hodge–Tate period morphism as in the following proposition.

Proposition 3.1. *The Hodge–Tate periods induce an equivalence*

$$\widehat{\mathrm{BT}}^{\diamond,\dagger}_{1,h,\eta} \xrightarrow{\sim} [\underline{\mathrm{GL}_h(\mathbb{Z}_p)} \backslash \mathbb{P}^{h-1,\diamond}_{\mathbb{Q}_p}].$$

Proof. Essential surjectivity can be reduced to the case where $S = \mathrm{Spa}(A, A^+)$ is strictly totally disconnected ([Sch17]) and one has a morphism

$$u : \underline{\mathbb{Z}_p^h} \to A$$

that induces a nonzero morphism

$$u_s : \mathbb{Z}_p^h \to K(s)$$

for all $s \in S$. Following Scholze–Weinstein ([SW12]), for all $s \in S$ we can find a p-divisible group H_s over $K(s)^\circ$ as well as a basis for $T_p(H_s^D)$, of Hodge–Tate period u_s. Using Lemma 2.5 we obtain a p-divisible group H over $\prod_{s \in S} K(s)^\circ$ equipped with a morphism

$$(\mathbb{Q}_p/\mathbb{Z}_p)^h \to H^D$$

and such that ω_H can be identified with $\prod_{s \in S} K(s)^\circ$. Let us denote by

$$v : \underline{\mathbb{Z}_p}^h \to (\prod_{s \in S} K(s)^\circ)[\frac{1}{\varpi}]$$

the morphism associated by the Hodge–Tate periods over

$$T = \mathrm{Spa}((\prod_{s \in S} K(s)^\circ)[\frac{1}{\varpi}], \prod_{s \in S} K(s)^\circ)$$

that lives over S. It coincides with the base change from S to T of u. The fully faithfulness of the Hodge–Tate period functor can then be deduced from point (2) of Lemma 2.5. □

Remark 3.2. We have only considered the case of p-divisible groups of dimension one, but in fact the previous isomorphism can be generalised to an isomorphism

$$\widehat{\mathrm{BT}}^{\diamond,\dagger}_{d,h,\eta} \xrightarrow{\sim} [\underline{\mathrm{GL}_h(\mathbb{Z}_p)} \backslash \mathrm{Gr}^{\diamond}_{d,h,\mathbb{Q}_p}],$$

where $\mathrm{Gr}_{d,h}$ denotes the Grassmannian of d-dimensional quotients of $\mathcal{O}^h$.

4 Height stratification

If H is a p-divisible group over $\mathcal{O}_C$ for $C/\mathbb{Q}_p$ complete and algebraically closed, the Hodge–Tate period

$$\alpha_{H^D} : T_p(H^D) \to \omega_H$$

gives us

$$T_p((H^{\text{ét}})^D) = \ker(\alpha_{H^D}),$$

where

$$0 \to H^\circ \to H \to H^{\text{ét}} \to 0$$

is the usual connected-étale decomposition of H. From this, one can easily deduce the following proposition.

Proposition 4.1. *For $1 \leq i \leq h$, the inverse image under π_h of the stratum of height i in $\mathbb{P}^{h-1,\diamond}_{\mathbb{Q}_p}$ can be identified with*

$$\Omega^{\diamond}_{i-1} \overset{P_i(\mathbb{Q}_p)}{\times} \underline{\mathrm{GL}_h(\mathbb{Q}_p)} \subset \mathbb{P}^{h-1,\diamond}_{\mathbb{Q}_p},$$

where

1. *$\Omega_{i-1} \subset \mathbb{P}^{i-1}_{\mathbb{Q}_p}$ is the Drinfeld space of dimension $i-1$, and P_i is the parabolic subgroup of GL_h formed by matrices*

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

with $A \in \mathrm{GL}_i$, and $B \in \mathrm{GL}_{h-i}$.

2. *The embedding in $\mathbb{P}^{h-1,\diamond}_{\mathbb{Q}_p}$ is given by a function that to any morphism*

$$u \colon \underline{\mathbb{Q}^i_p} \to \mathcal{L}$$

in $\Omega^{\diamond}_{i-1}$ associates

$$u \oplus 0 \colon \underline{\mathbb{Q}^i_p} \oplus \underline{\mathbb{Q}^{h-i}_p} \to \mathcal{L}.$$

3. *The image consists of the locally closed subset in $|\mathbb{P}^{h-1}_{\mathbb{Q}_p}|$ given by the $x \in |\mathbb{P}^{h-1}_{\mathbb{Q}_p}|$ such that*

$$\dim_{\mathbb{Q}_p} \ker(\mathbb{Q}^h_p \to K(x)) = h - i.$$

4. *In particular, the open stratum corresponding to $i = h$ is $\Omega^{\diamond}_{h-1}$.*

5 Relation to Lubin–Tate spaces

The link between Proposition 1.1.4 and point (4) of Proposition 4.1 follows from the isomorphism

$$\underline{\mathrm{GL}_h(\mathbb{Z}_p)} \backslash \Omega^{\diamond}_{h-1} \simeq [J^{\mathrm{ad}}_h \backslash \mathfrak{X}^{\diamond}_{h,\eta}]$$

between the twin-towers, where J^{ad}_h is a v-group on $\mathrm{Perf}_{\mathbb{F}_p}$, a twisted form of $\underline{\mathcal{O}^{\times}_D}$. We refer to [CFS17, Section 5.1] for the general form of the isomorphism between the two towers for any group G.

6 A rigid analytic presentation of $\widehat{\mathcal{M}}^{\leq h,\diamond,\dagger}_{\mathrm{FG},p,\eta}$

Let us take the following definition.

Definition 6.1. Let us denote by $\mathcal{P}$ the groupoid over $\mathbb{P}^{h-1,\diamond}_{\mathbb{Q}_p}$ whose objects over $S \to \mathbb{P}^{h-1,\diamond}_{\mathbb{Q}_p}$ associated to a morphism

$$u \colon \underline{\mathbb{Q}^h_p}_S \to \mathcal{L}$$

with $\mathcal{L}$ a locally free $\mathcal{O}_{S^\sharp}$-module of rank one are pairs (g, f) where

- $g \in \underline{\mathrm{GL}_h(\mathbb{Z}_p)}(S)$

- f is an isomorphism

$$f\colon \underline{\mathbb{Q}_p^h}_S/\ker(u) \xrightarrow{\sim} \underline{\mathbb{Q}_p^h}_S/\ker(u\circ g)$$

making the diagram

$$\begin{array}{ccc} \underline{\mathbb{Q}_p^h}_S/\ker(u) & & \\ {\scriptstyle f}\downarrow{\scriptstyle \sim} & \overset{u}{\searrow} & \\ & & \mathcal{L} \\ & \underset{u\circ g}{\nearrow} & \\ \underline{\mathbb{Q}_p^h}_S/\ker(u\circ g) & & \end{array}$$

commute.

The proof of the theorem is left to the reader.

Theorem 6.2. *There is an isomorphism*

$$\mathcal{P} \xrightarrow{\sim} \mathbb{P}^{h-1,\diamond}_{\mathbb{Q}_p} \times_{\widehat{\mathcal{M}}^{\leq h,\diamond,\dagger}_{\mathrm{FG},p,h}} \mathbb{P}^{h-1,\diamond}_{\mathbb{Q}_p},$$

whence we obtain a presentation in the v-topology

$$\mathcal{P} \rightrightarrows \mathbb{P}^{h-1,\diamond}_{\mathbb{Q}_p} \to \widehat{\mathcal{M}}^{\leq h,\diamond,\dagger}_{\mathrm{FG},p,\eta}.$$

7 Coherent/quasicoherent coefficients

7.1 Coherent coefficients

We must be careful since $\mathrm{Spf}(\mathbb{Z}_p\langle v_1, v_2, \ldots\rangle)$ is not a Noetherian formal scheme, so there is a priori no good notion of coherent sheaves on $\widehat{\mathcal{M}}_{\mathrm{FG},p}$. Per definition, a coherent sheaf on $\mathcal{M}_{\mathrm{FG},p}$ is a quasicoherent sheaf such that for all morphisms $\mathrm{Spec}(A) \to \mathcal{M}_{\mathrm{FG},(p)}$, the associated A-module is of finite type.

Proposition 7.1. *The sheaf of $\mathcal{O}_v^+$-modules on $\mathbb{P}^{h-1,\diamond}_{\mathbb{Q}_p}$ associated to a coherent sheaf on $\mathcal{M}_{\mathrm{FG},(p)}$ is a perfect complex.*

Proof. We use Lemma 7.2. In the notation of this lemma, since X is a regular scheme, the associated coherent sheaf on X is a perfect complex. □

Lemma 7.2. *There exists a proper smooth scheme X over $\mathrm{Spec}(\mathbb{Z}_{(p)})$ equipped with a morphism $X \to \mathrm{BT}_{1,h}$ such that the associated morphism on p-adic completions*

$$\widehat{X} \to \widehat{\mathrm{BT}}_{1,h}$$

is formally étale and the morphism induced by the Hodge–Tate periods

$$\widehat{X}^{\diamond}_{\eta} \to [\underline{\mathrm{GL}_h(\mathbb{Z}_p)}\backslash \mathbb{P}^{h-1,\diamond}_{\mathbb{Q}_p}]$$

is a v-cover.

Proof. It suffices to choose for X an integral model of a Shimura variety of the type used by Harris and Taylor ([HT01]) for a compact hyperspecial level at p. □

Remark 7.3. We could equally well have used [CS69, Theorem 1.6] with a stronger notion of coherent sheaves requiring that the pullback to $\mathrm{Spec}(\mathbb{Z}_{(p)}[v_k]_{k\geq 1})$ be given by a coherent module. This is the case of interest to us following [CS69, Theorem 1.6].

Remark 7.4. The point of view of Shimura varieties of the type considered in Harris–Taylor is used in [BL10].

Corollary 7.5. *For every prime number p and and integer $h \geq 1$, we have a functor*

$$\mathrm{SH}_{\mathrm{fin}} \to \mathrm{Perf}_{\mathcal{O}_v^{\sharp,+}}([\underline{\mathrm{GL}_h(\mathbb{Z}_p)}\backslash \mathbb{P}^{h-1,\diamond}_{\mathbb{Q}_p}]).$$

from the stable homotopy category of finite spectra to the category of $\underline{\mathrm{GL}_h(\mathbb{Z}_p)}$-equivariant perfect complexes of $\mathcal{O}_v^{\sharp,+}$-modules on $\mathbb{P}^{h-1,\diamond}_{\mathbb{Q}_p}$.

Upon inverting p, we can hope to use recent results in p-adic Simpson theory ([Heu23], [AM24]) to find a functor

$$\mathrm{SH}_{\mathrm{fin}}[\frac{1}{p}] \xrightarrow{?} \{\underline{\mathrm{GL}_h(\mathbb{Z}_p) \times \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}\text{-equivariant Higgs bundles on } \mathbb{P}^{h-1}_{\mathbb{C}_p}\}.$$

Remark 7.6. When we speak about projective space in this context, we are free to choose between the adic space version or the algebraic version according to GAGA. One of the fascinating aspects of such a construction is that it would produce algebraic Higgs bundles on $\mathbb{P}^{h-1}_{\mathbb{C}_p}$ starting from finite spectra and passing through a rigid analytic detour.

7.2 Quasicoherent coefficients

In the case where our sheaf on $\mathcal{M}_{\mathrm{FG},(p)}$ is only quasicoherent we still have a *geometric* structure on the associated object over $[\underline{\mathrm{GL}_h(\mathbb{Z}_p)}\backslash \mathbb{P}^{h-1,\diamond}_{\mathbb{Q}_p}]$. For this, we use the work of Mann [Man22]. We hence obtain the following result.

Corollary 7.7. *There is a natural functor*

$$\mathrm{SH} \to \textit{Complexes of quasicoherent } \mathcal{O}^{\sharp,+,a}\textit{-modules à la Mann on } [\underline{\mathrm{GL}_h(\mathbb{Z}_p)}\backslash \mathbb{P}^{h-1,\diamond}_{\mathbb{Q}_p}].$$

The objects defined by Mann may seem abstract but they are not thanks to the descent theorem [Man22, Theorem 3.1.1.7]. In fact, if we consider the perfectoid projective space

$$\mathbb{P}^{h-1,1/p^\infty}_{\mathbb{C}_p} := \varprojlim_{n\geq 1} \mathbb{P}^{h-1}_{\mathbb{C}_p},$$

where the transition maps are given by $[x_1 : \cdots : x_h] \mapsto [x_1^p : \cdots : x_h^p]$, the morphism

$$\mathbb{P}^{h-1,1/p^\infty,\flat}_{\mathbb{C}_p} \to \mathbb{P}^{h-1,\diamond}_{\mathbb{Q}_p} \tag{1}$$

is a quasi-pro-étale cover ([Sch17]) which is pro-étale on the open

$$\bigcup_{i=1}^{h} x_i \neq 0$$

complement of h-hyperplanes (which contains the Drinfeld space Ω_{h-1}). Therefore, if for $1 \leq i \leq h$ we have,

$$U_i = \{[x_1 : \cdots : x_h] \mid |x_j| \leq |x_i| \neq 0 \text{ for all } 1 \leq j \leq h\} \subset \mathbb{P}^{h-1,1/p^\infty}_{\mathbb{C}_p}$$

we have

$$U_i = \mathrm{Spa}(\mathbb{C}_p\langle(\frac{x_1}{x_i})^{1/p^\infty}, \ldots, (\frac{x_h}{x_i})^{1/p^\infty}\rangle, \mathcal{O}_{\mathbb{C}_p}\langle(\frac{x_1}{x_i})^{1/p^\infty}, \ldots, (\frac{x_h}{x_i})^{1/p^\infty}\rangle),$$

a perfectoid closed ball of dimension $h-1$ over $\mathbb{C}_p$. On the quasi-pro-étale cover (1) these complexes are given by the objects of

$$\varprojlim_{[n]\in\Delta} \prod_{1\leq i_1,\ldots,i_n\leq h} \mathcal{D}(\mathcal{O}(U_{i_1} \cap \cdots \cap U_{i_n})^{+,a}),$$

where Δ is the simplicial category, and $\mathcal{D}(\mathcal{O}(U_{i_1} \cap \cdots \cap U_{i_n})^{+,a})$ is the usual stable ∞-category of $\mathcal{O}(U_{i_1} \cap \cdots \cap U_{i_n})^{+,a}$-modules.

8 The non-overconvergent case

In this text we have only considered the overconvergent versions of our stacks, i.e. p-divisible groups/formal groups over R° and not any R^+. This leads to the following questions.
Question:

1. Let (C, C^+) be a complete extension of $(\mathbb{Q}_p, \mathbb{Z}_p)$ with C algebraically closed and $\mathcal{G}$ a p-divisible formal group over C^+. IS there a p-divisible group H over C^+ such that $\widehat{H} \simeq \mathcal{G}$?
2. Can we extend the classification of Scholze–Weinstein of p-divisible groups over $\mathcal{O}_C = C^\circ$ ([SW12]) to a classification of p-divisible groups over C^+?

The author does not know the answer to these questions.

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